Differential Equations MA2042 Midterm Exam 2

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Problem 1. (a) (15%) Let $f : [0,\infty) \to \mathbb{R}$ be piecewise continuous and of exponential order a for some $a \in \mathbb{R}$, and $F = \mathscr{L}(f)$ be the Laplace transform of f. Assuming the dominated convergence theorem (in which the integrability is equivalent to the existence of the improper integral)

> Let $f_n : [0,\infty) \to \mathbb{R}$ be a sequence of integrable functions such that $\{f_n\}_{n=1}^\infty$ converges pointwise to some integrable function f on $[0,\infty)$. Suppose that there is an integrable function g such that $|f_n(x)| \leqslant g(x)\ \forall\, x \in [0,\infty)$. Then $\displaystyle\lim_{n\to\infty}\int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx$.

show that $F^{(n)}(s) = \mathscr{L}(g_n)(s)$ for $s > a$, where $g_n(t) = (-t)^n f(t)$.

(b) (15%) Let $f : [0,\infty)$ be piecewise continuous and of exponential order a for some $a \in \mathbb{R}$ such that $\displaystyle\lim_{t\to 0^+}\frac{f(t)}{t}$ exists. Show that

$$\mathscr{L}\Big(\frac{f(t)}{t}\Big)(s) = \int_s^\infty F(y)\,dy\,,$$

where $F = \mathscr{L}(f)$ is the Laplace transform of f. $\Big($**Hint**: *Consider* $\dfrac{d}{ds}\mathscr{L}\Big(\dfrac{f(t)}{t}\Big)(s)\Big)$

(c) (10%) Using part (a) to find the inverse Laplace transform of $F(s) = \ln\dfrac{s+2}{s-5}$.

(d) (10%) Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0\,.$$

Let $Y = \mathscr{L}(y)$ be the Laplace transform of a solution y. Show that Y satisfies

$$(1+s^2)Y'(s) + sY(s) = 0\,.$$

Proof. (a) First we note that

$$1 - t \leqslant e^{-t} \leqslant 1 - t + \frac{t^2}{2} \qquad \forall\, t \in \mathbb{R}\,;$$

thus

$$-\frac{h}{|h|}t \leqslant \frac{e^{-ht}-1}{|h|} \leqslant -\frac{h}{|h|}t + \frac{|h|t^2}{2} \qquad \forall\, h \in \mathbb{R} \text{ and } t > 0\,.$$

Therefore,

$$\Big|\frac{e^{-(s+h)t} - e^{-st}}{h}\Big| = e^{-st}\Big|\frac{e^{-ht}-1}{h}\Big| \leqslant \Big(t + \frac{t^2}{2}\Big)e^{-st} \qquad \forall\, |h| \leqslant 1 \text{ and } t > 0\,.$$

Now, since f is piecewise continuous and of exponential order a, there exists $M > 0$ such that $|f(t)| \leqslant Me^{at}$ for all $t > 0$. Let $g(t) = Me^{(a-s)t}\big(t + \frac{t^2}{2}\big)$. Then for $s > a$, g is integrable (that is, $\int_0^\infty g(t)\,dt < \infty$) and $\left|\frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\right| \leqslant g(t)$; thus the dominated convergence theorem implies that for $s > a$,

$$\begin{aligned} F'(s) &= \lim_{h\to 0} \int_0^\infty \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\,dt = \int_0^\infty \lim_{h\to 0} \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\,dt = \int_0^t \frac{\partial}{\partial s} e^{-st} f(t)\,dt \\ &= \int_0^t (-t)e^{-st} f(t)\,dt = \mathscr{L}(-tf(t))(s) = \mathscr{L}(g_1)(s)\,. \end{aligned}$$

Moreover, g_1 is of exponential order b as long as $b > a$; thus for $s > a$, $s > b$ for some $b > a$ and using what we just established we find that

$$\frac{d^2}{ds^2} F(s) = \frac{d}{ds}\mathscr{L}(g_1)(s) = \mathscr{L}(-tg_1(t))(s) = \mathscr{L}(g_2)(s)\,.$$

By induction, we conclude that $F^{(n)}(s) = \mathscr{L}(g_n)(s)$ for $s > a$.

(b) Since $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists and f is a piecewise continuous function of exponential order a, the function $g(t) = \frac{f(t)}{t}$ is also piecewise continuous and of exponential order a. Therefore, the Laplace transform of g exists and using (a),

$$\frac{d}{ds}\mathscr{L}(g)(s) = \mathscr{L}(-f)(s) = -F(s)\,.$$

Therefore, $\mathscr{L}\big(\frac{f(t)}{t}\big)(s) = \mathscr{L}\big(\frac{f(t)}{t}\big)(a) - \int_s^a F(y)\,dy$.

(c) Suppose that $\mathscr{L}(f) = F$. Since $F'(s) = \frac{1}{s+2} - \frac{1}{s-5}$, using (a) we find that

$$\mathscr{L}\big(-tf(t)\big)(s) = F'(s) = \frac{1}{s+2} - \frac{1}{s-5} = \mathscr{L}(e^{-2t})(s) - \mathscr{L}(e^{5t})(s)\,;$$

thus $f(t) = \frac{e^{5t} - e^{-2t}}{t}$.

(d) Using (a),

$$\begin{aligned} \mathscr{L}(ty'' + y' + ty)(s) &= -\frac{d}{ds}\mathscr{L}(y'')(s) + \mathscr{L}(y')(s) - \frac{d}{ds}\mathscr{L}(y)(s) \\ &= -\frac{d}{ds}\big[s^2 Y(s) - sy(0) - y'(0)\big] + \big[sY(s) - y(0)\big] - Y'(s) \\ &= -(s^2+1)Y'(s) - 2sY(s) + sY(s) \end{aligned}$$

which implies that $(1+s^2)Y'(s) + sY(s) = 0$. □

Problem 2. (15%) Solve the ODE

$$y'' - 4y' + 4y = t^2 e^t\,, \qquad y(0) = y'(0) = 0$$

using the Laplace transform.

Solution: **(Method 1)**: Letting Y be the Laplace transform of y, we find that

$$Y(s) = \frac{1}{s^2 - 4s + 4}\mathscr{L}(t^2 e^t)(s) = \frac{1}{(s-2)^2}\mathscr{L}(t^2 e^t)(s)\,;$$

thus by the fact that $\mathscr{L}^{-1}\big(\frac{1}{(s-2)^2}\big)(t) = te^{2t}$,

$$y(t) = \big(te^{2t}\big) * \big(t^2 e^t\big)(t) = \int_0^t s^2 e^s (t-s) e^{2(t-s)}\, ds = te^{2t}\int_0^t s^2 e^{-s}\, ds - e^{2t}\int_0^t s^3 e^{-s}\, ds\,.$$

Since

$$\begin{aligned}\int_0^t s^2 e^{-s}\, ds &= -s^2 e^{-s}\Big|_{s=0}^{s=t} + 2\int_0^t s e^{-s}\, ds = -t^2 e^{-t} + 2\Big[-se^{-s}\Big|_{s=0}^{s=t} + \int_0^t e^{-s}\, ds\Big] \\ &= -t^2 e^{-t} - 2te^{-t} + 2(1 - e^{-t}) = -(t^2 + 2t + 2)e^{-t} + 2\,,\end{aligned}$$

we conclude that

$$\begin{aligned}y(t) &= te^{2t}\int_0^t s^2 e^{-s}\, ds - e^{2t}\Big[-s^3 e^{-s}\Big|_{s=0}^{s=t} + 3\int_0^t s^2 e^{-s}\, ds\Big] \\ &= -te^{2t}(t^2 + 2t + 2)e^{-t} + 2te^{2t} + e^{2t}t^3 e^{-t} + 3e^{2t}(t^2 + 2t + 2)e^{-t} - 6e^{2t} \\ &= (t^2 + 4t + 6)e^t + 2(t-3)e^{2t}\,.\end{aligned}$$

(Method 2): Letting Y be the Laplace transform of y, we find that

$$Y(s) = \frac{1}{s^2 - 4s + 4}\mathscr{L}(t^2 e^t)(s) = \frac{2}{(s-2)^2(s-1)^3}\,.$$

Using partial fraction,

$$\frac{2}{(s-2)^2(s-1)^3} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3}\,,$$

where A, B, C, D, E satisfy

$$A(s-2)(s-1)^3 + B(s-1)^3 + C(s-2)^2(s-1)^2 + D(s-1)(s-2)^2 + E(s-2)^2 = 2\,.$$

Therefore, $A = -6$, $B = 2$, $C = 6$, $D = 4$ and $E = 2$; thus

$$y(t) = -6e^{2t} + 2te^{2t} + 6e^t + 4te^t + t^2 e^t = (t^2 + 4t + 6)e^t + 2(t-3)e^{2t}\,.$$

□

Problem 3. (15%) Solve the ODE

$$y'' + 5y' + 6y = e^{-t}\delta_2(t) = e^{-t}\delta(t-2)\,, \qquad y(0) = 2, \quad y'(0) = -5\,.$$

Hint: *You can treat* $\delta_c(t) = \delta(t-c)$ *as a normal function whose Laplace transform is* e^{-cs} *and apply the properties of the Laplace transforms to compute the Laplace transform of* $e^{-t}\delta_2(t)$.

Solution: Letting Y be the Laplace transform of y, we find that

$$\begin{aligned} Y(s) &= \frac{2(s+5)-5}{s^2+5s+6} + \frac{e^{-2(s+1)}}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)} + e^{-2}\Big[\frac{e^{-2s}}{s+2} - \frac{e^{-2s}}{s+3}\Big] \\ &= \frac{1}{s+2} + \frac{1}{s+3} + e^{-2}\Big[\frac{e^{-2s}}{s+2} - \frac{e^{-2s}}{s+3}\Big]\,. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= e^{-2t} + e^{-3t} + e^{-2}\big[u_2(t)e^{-2(t-2)} - u_2(t)e^{-3(t-2)}\big] \\ &= e^{-2t} + e^{-3t} + u_2(t)\big(e^{-2t+2} - e^{-3t+4}\big)\,. \end{aligned}$$

□

Problem 4. The ***impulse response function*** for the ODE $y'' + by' + cy = g(t)$ is the function h whose Laplace transform is given by

$$\mathscr{L}(h)(s) = \frac{Y(s)}{G(s)},$$

where Y is the Laplace transform of the solution to $y'' + by' + cy = g(t)$ with initial condition $y(0) = y'(0) = 0$, and $G = \mathscr{L}(g)$.

(a) (5%) Show that h is independent of g.

(b) (10%) Show that $h(t) \to 0$ as $t \to \infty$ if and only if the real part of the roots to $r^2 + br + c = 0$ are strictly less than zero.

Proof. (a) If Y is the Laplace transform of $y'' + by' + cy = g(t)$ with initial data $y(0) = y'(0) = 0$ and G is the Laplace transform of g, then

$$Y(s) = \frac{G(s)}{s^2 + bs + c}.$$

Therefore, $\mathscr{L}(h)(s) = \dfrac{Y(s)}{G(s)} = \dfrac{1}{s^2 + bs + c}$ which implies that $\mathscr{L}(h)(s)$ is independent of g.

(b) (1) if $r^2 + br + c = 0$ has two distinct real roots r_1 and r_2, then

$$\mathscr{L}(h)(s) = \frac{1}{(s - r_1)(s - r_2)} = \frac{1}{r_1 - r_2}\left[\frac{1}{s - r_1} - \frac{1}{s - r_2}\right];$$

thus $h(t) = \dfrac{1}{r_1 - r_2}\left[e^{r_1 t} - e^{r_2 t}\right]$ which implies that $h(t) \to 0$ as $t \to \infty$ if and only if $r_1, r_2 < 0$.

(2) if $r^2 + br + c = 0$ has a double root r_1, then

$$\mathscr{L}(h)(s) = \frac{1}{(s - r_1)^2};$$

thus $h(t) = te^{r_1 t}$ which implies that $h(t) \to 0$ as $t \to \infty$ if and only if $r_1 < 0$.

(3) if $r^2 + br + c = 0$ has two complex roots $\lambda \pm i\mu$, then

$$\mathscr{L}(h)(s) = \frac{1}{(s - \lambda)^2 + \mu^2} = \frac{1}{\mu}\frac{\mu}{(s - \lambda)^2 + \mu^2};$$

thus $h(t) = \dfrac{1}{\mu}e^{\lambda t}\sin \mu t$ which implies that $h(t) \to 0$ as $t \to \infty$ if and only if $\lambda < 0$.

In either cases, $h(t) \to 0$ as $t \to \infty$ if and only if the real part of the roots to $r^2 + br + c = 0$ are strictly less than zero. $\square$

Problem 5. (15%) Solve the integro-differential equation

$$\phi'(t) + \phi(t) = \int_0^t \sin(t-\xi)\phi(\xi)\,d\xi \qquad \phi(0) = 1$$

using the Laplace transform.

Solution: Taking the Laplace transform of the integro-differential equation, we find that

$$s\mathscr{L}(\phi)(s) - \phi(0) + \mathscr{L}(\phi)(s) = \mathscr{L}(\sin t)(s)\mathscr{L}(\phi)(s)$$

which implies that

$$(s+1)\mathscr{L}(\phi)(s) - 1 = \frac{\mathscr{L}(\phi)}{s^2+1}\,.$$

Therefore,

$$\big[(s^2+1)(s+1) - 1\big]\mathscr{L}(\phi)(s) = s^2 + 1$$

or equivalently,

$$\mathscr{L}(\phi)(s) = \frac{s^2+1}{s^3+s^2+s} = \frac{1}{s} - \frac{1}{s^2+s+1} = \frac{1}{s} - \frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\,.$$

Taking the inverse Laplace transform, we conclude that

$$\phi(t) = 1 - \frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}t}{2}\,.$$

□