Differential Equations MA2042 Final Exam

National Central University, Jun. 15 2016

Problem 1. (20%) Solve the boundary value problem

$$x^2y'' - 2xy' + 2y = 0\,, \qquad y(1) = -2\,, \quad y(2) = 2\,.$$

Solution: Solving the "characteristic" equation $r(r-1) - 2r + 2 = 0$, we find that $r = 1$ or $r = 2$; thus the general solution to $x^2y'' - 2xy' + 2y = 0$ can be written as

$$y(x) = C_1x + C_2x^2\,.$$

Making use of the boundary condition, we have

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix};$$

thus $(C_1, C_2) = (-5, 3)$. Therefore, the solution to the given boundary value problem is $y(x) = -5x^2 + 3x$. □

Problem 2. (20%) Let $f : [-L, L] \to \mathbb{R}$ be a function such that f, f', f'' are piecewise continuous on $[-L, L]$. Show that if $\{c_n\}_{n=0}^{\infty}, \{s_n\}_{n=1}^{\infty}$ are the Fourier coefficients of f; that is,

$$c_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx \quad \text{and} \quad s_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx\,,$$

then the sequences $\{n^2c_n\}_{n=0}^{\infty}$ and $\{n^2s_n\}_{n=1}^{\infty}$ are bounded.

Proof. We only show that $\{n^2c_n\}_{n=1}^{\infty}$ is bounded since the proof of the other case is similar. Let $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ be the Fourier coefficients of f''; that is,

$$s(f'', x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\Big(a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\Big)\,,$$

where $a_k = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f''(x)\cos\frac{k\pi x}{L}\,dx$ and $b_k = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f''(x)\sin\frac{k\pi x}{L}\,dx$. Integrating by parts,

$$\begin{aligned}
a_k &= \frac{1}{L}\Big[f'(x)\cos\frac{k\pi x}{L}\Big|_{x=-L}^{x=L} + \frac{k\pi}{L}\int_{-L}^{L} f'(x)\sin\frac{k\pi x}{L}\,dx\Big] \\
&= \frac{1}{L}\Big[\big(f'(L) - f'(-L)\big)(-1)^k + \frac{k\pi}{L}\Big(f(x)\sin\frac{k\pi x}{L}\Big|_{x=-L}^{x=L} - \frac{k\pi}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx\Big)\Big] \\
&= (-1)^k\frac{f'(L) - f'(-L)}{L} - \frac{k^2\pi^2}{L^2}\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx \\
&= (-1)^k\frac{f'(L) - f'(-L)}{L} - \frac{k^2\pi^2}{L^2}c_k\,.
\end{aligned}$$

Note that by the Parseval identity, $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=1}^\infty$ satisfy

$$L\Big[\frac{a_0^2}{2}+\sum_{k=1}^\infty (a_k^2+b_k^2)\Big] = \int_{-L}^{L} \big|f''(x)\big|^2\,dx < \infty\,;$$

thus $\{a_k\}_{k=1}^\infty$, $\{b_k\}_{k=1}^\infty$ are bounded. Therefore, there exists $M>0$ such that

$$\Big|(-1)^k\frac{f'(L)-f'(-L)}{L} - \frac{k^2\pi^2}{L^2}c_k\Big| = |a_k| \leqslant M \qquad \forall\, k\in\mathbb{N}\,;$$

thus for some $C>0$,

$$\Big|\frac{k^2\pi^2}{L^2}c_k\Big| \leqslant M + \Big|(-1)^k\frac{f'(L)-f'(-L)}{L}\Big| \leqslant C \qquad \forall\, k\in\mathbb{N}\,.$$

This implies that $\{n^2c_n\}_{n=1}^\infty$ is bounded. □

Problem 3. (20%) Prove that

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots = \sum_{n=0}^\infty \frac{1}{(2n+1)^2}$$

by applying the Parseval identity to periodic extension of the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi<x<0\,,\\ \pi & \text{if } 0<x<\pi\,.\end{cases}$$

Proof. We compute Fourier coefficients as follows:

$$c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \pi\,,$$

and if $k\in\mathbb{N}$,

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \int_0^\pi \cos kx\,dx = \frac{\sin kx}{k}\Big|_{x=0}^{x=\pi} = 0\,,$$

Moreover, for $k\in\mathbb{N}$,

$$s_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \int_0^\pi \sin kx\,dx = -\frac{\cos kx}{k}\Big|_{x=0}^{x=\pi} = \frac{1-(-1)^k}{k}\,.$$

Therefore, the Fourier series representation of f is

$$s(f,x) = \frac{\pi}{2}+\sum_{k=1}^\infty \frac{1-(-1)^k}{k}\sin kx = \frac{\pi}{2}+\sum_{k=0}^\infty \frac{2}{2k+1}\sin(2k+1)x\,.$$

Therefore, the Parseval identity implies that

$$\pi^3 = \int_{-\pi}^{\pi}\big|f(x)\big|^2\,dx = \pi\Big[\frac{c_0^2}{2}+\sum_{k=0}^\infty s_k^2\Big] = \pi\Big[\frac{\pi^2}{2}+4\sum_{k=0}^\infty\frac{1}{(2k+1)^2}\Big]\,;$$

thus $\displaystyle\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{1}{4}\Big[\pi^2-\frac{\pi^2}{2}\Big] = \frac{\pi^2}{8}$. □

Problem 4. (20%) Solve the equation

$$\begin{aligned} u_t(x,t) + u(x,t) &= u_{xx}(x,t) && 0 < x < \pi \text{ and } t > 0\,, \\ u(x,0) &= \sin x && 0 < x < \pi\,, \\ u(0,t) = u(\pi,t) &= 0 && t > 0 \end{aligned}$$

by solving the heat equation for w, where $w(x,t) = e^t u(x,t)$.

Solution: If $w(x,t) = e^t u(x,t)$, then w satisfies

$$\begin{aligned} w_t(x,t) &= w_{xx}(x,t) && 0 < x < \pi \text{ and } t > 0\,, \\ w(x,0) &= \sin x && 0 < x < \pi\,, \\ w(0,t) = w(\pi,t) &= 0 && t > 0\,. \end{aligned}$$

Therefore, using the representation formula for the heat equation, we find that

$$w(x,t) = \sum_{k=1}^{\infty} s_k e^{-\frac{k^2\pi^2 t}{\pi^2}} \sin \frac{k\pi x}{\pi} = \sum_{k=1}^{\infty} s_k e^{-k^2 t} \sin kx\,,$$

where $s_k = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} \sin x \sin kx \, dx = \delta_{k1}$. In other words,

$$w(x,t) = \sum_{k=1}^{\infty} \delta_{k1} e^{-k^2 t} \sin kx = e^{-t} \sin x\,;$$

thus $u(x,t) = e^{-t} w(x,t) = e^{-2t} \sin x$. □

Problem 5. (20%) Find the formal solution to the dispersive wave equations

$$\begin{aligned} u_{tt}(x,t) + u(x,t) &= u_{xx}(x,t) && 0 < x < \pi \text{ and } t > 0\,, \\ u(x,0) = f(x)\,, \quad u_t(x,0) &= 0 && 0 < x < \pi\,, \\ u(0,t) = u(\pi,t) &= 0 && t > 0 \end{aligned}$$

using the method of separation of variables.

Solution: Suppose that $u(x,t) = X(x)T(t)$ is a solution to

$$\begin{aligned} u_{tt}(x,t) + u(x,t) &= u_{xx}(x,t) && 0 < x < \pi \text{ and } t > 0\,, \\ u(0,t) = u(\pi,t) &= 0 && t > 0\,. \end{aligned}$$

Then

$$\begin{aligned} \big(T''(t) + T(t)\big) X(x) &= T(t) X''(x) && 0 < x < \pi \text{ and } t > 0\,, \\ X(0) = X(\pi) &= 0 && t > 0\,; \end{aligned}$$

thus

$$\frac{T''(t) + T(t)}{T(t)} = \frac{X''(x)}{X(t)} = -\lambda \qquad 0 < x < \pi \text{ and } t > 0$$

for some constant λ. To have non-trivial solution, $\lambda > 0$. Moreover, for $k \in \mathbb{N}$, $\lambda = \lambda_k = k^2$ and $X_k(x) = \sin kx$. The corresponding $T_k(t)$ satisfies

$$T_k'' + (k^2+1)T(t) = 0\,;$$

thus $T_k(t) = c_k \cos\sqrt{k^2+1}t + s_k \sin\sqrt{k^2+1}t$. Then we look for solution $u(x,t)$ of the form

$$u(x,t) = \sum_{k=1}^{\infty} \big(c_k \cos\sqrt{k^2+1}\,t + s_k \sin\sqrt{k^2+1}\,t\big) \sin kx\,.$$

To satisfy the initial condition, we need

$$\sum_{k=1}^{\infty} c_k \sin kx = f(x) \quad \text{and} \quad \sum_{k=1}^{\infty} s_k\sqrt{k^2+1}\sin kx = 0\,.$$

Therefore, $s_k = 0$ for all $k \in \mathbb{N}$, and $c_k = \dfrac{2}{\pi}\displaystyle\int_0^\pi f(x)\sin kx\,dx$; thus the (formal) solution to the wave equation given above is

$$u(x,t) = \frac{2}{\pi}\sum_{k=1}^{\infty}\Big(\int_0^\pi f(x)\sin kx\,dx\Big)\cos\sqrt{k^2+1}\,t\sin kx\,.$$

Problem 6. (20%) Let u be a smooth solution to the wave equation

$$\begin{aligned} \rho u_{tt}(x,t) &= T u_{xx}(x,t) && 0 < x < L \text{ and } t > 0\,,\\ u(x,0) = f(x)\,,\quad u_t(x,0) &= 0 && 0 < x < L\,,\\ u(0,t) = u(L,t) &= 0 && t > 0\,. \end{aligned}$$

Show that the energy $E(t) = \displaystyle\int_0^L \Big[\frac{\rho}{2}\big|u_t(x,t)\big|^2 + \frac{T}{2}\big|u_x(x,t)\big|^2\Big]\,dx$ is independent of t; that is, $E'(t) = 0$ for all $t > 0$.

Proof. Differentiating $E(t)$ in t, integrating by parts we find that

$$\begin{aligned} E'(t) &= \frac{d}{dt}\int_0^L \Big[\frac{\rho}{2}\big|u_t(x,t)\big|^2 + \frac{T}{2}\big|u_x(x,t)\big|^2\Big]\,dx\\ &= \int_0^L \Big[\rho u_t(x,t)u_{tt}(x,t) + T u_x(x,t)u_{xt}(x,t)\Big]\,dx\\ &= \int_0^L \rho u_t(x,t)u_{tt}(x,t)\,dx + T\int_0^L u_x(x,t)u_{xt}(x,t)\,dx\\ &= \int_0^L \rho u_t(x,t)u_{tt}(x,t)\,dx + T\Big[u_x(x,t)u_t(x,t)\Big|_{x=0}^{x=L} - \int_0^L u_{xx}(x,t)u_t(x,t)\,dx\Big]\\ &= \int_0^L \big[\rho u_{tt}(x,t) - T u_{xx}(x,t)\big]u_t(x,t)\,dx + T u_x(x,t)u_t(x,t)\Big|_{x=0}^{x=L}\,. \end{aligned}$$

Since $u(0,t) = u(L,t) = 0$ for all $t > 0$, we have $u_t(0,t) = u_t(L,t) = 0$ for all $t > 0$; thus the second term on the right-hand side of the equality above vanishes. Since u satisfies the wave equation, the first term on the right-hand side also vanishes, so we conclude that $E'(t) = 0$ for all $t > 0$. □