Differential Equations MA2041-A Midterm Exam 2

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1	2(1)	2(2)	3(1)	3(2)	4(1)	4(2)	5	Total

Problem 1. (15%) Use the method of reduction of order to find the general solution to the second order differential equation

$$(\sin t)y'' - (\sin t + \cos t)y' + (\cos t)y = 0 \qquad \text{where} \quad 0 < t < \pi\,, \tag{1}$$

provided that one solution $y = \varphi_1(t) = e^t$ is given.

Solution: Assume the solution to (1) can be written as $y = v(t)e^t$. Then

$$(\sin t)(v'' + 2v' + v)e^t - (\sin t + \cos t)(v' + v)e^t + (\cos t)ve^t = 0\,;$$

thus

$$(\sin t)v'' + \big[2\sin t - (\sin t + \cos t)\big]v' = 0\,.$$

Therefore,

$$v'' + \big(1 - \cot t\big)v' = 0\,.$$

Since $\displaystyle\int \cos t dt = \log \sin t$, we find that

$$\Big(\frac{e^t}{\sin t}v'\Big)' = 0\,.$$

Therefore, $v'(t) = C_1 e^{-t}\sin t$ which implies that

$$v(t) = C_1 \int e^{-t}\sin t dt = -\frac{C_1}{2}e^{-t}(\sin t + \cos t) + C_2\,.$$

As a consequence, the general solution to (1) is

$$y(t) = v(t)e^t = C_1(\sin t + \cos t) + C_2 e^t\,.$$

Problem 2. (1) (15%) Use the method of variation of parameters to show that

$$y(t) = C_1 \cos 3t + C_2 \sin 3t + \frac{1}{3}\int_0^t f(s)\sin 3(t-s)\, ds$$

is a general solution to the ordinary differential equation

$$y'' + 9y = f(t)\,. \tag{2}$$

(2) (15%) Find the solution to

$$y'' + 9y = 4\cos^3 t \tag{3}$$

that also satisfies the initial condition $y(0) = y'(0) = 0$.

Solution:

1. Note that $\{\cos 3t, \sin 3t\}$ is a fundamental set of the corresponding homogeneous equation $y'' + 9y = 0$. The method of variation of parameters implies that a particular solution to (2) is given by

$$\begin{aligned} y = Y(t) &= -\cos 3t \int_0^t \frac{\sin 3s f(s)}{W(\cos 3s, \sin 3s)}\, ds + \sin 3t \int_0^t \frac{\cos 3s f(s)}{W(\cos 3s, \sin 3s)}\, ds \\ &= \frac{1}{3}\int_0^t f(s)(\cos 3s \sin 3t - \cos 3t \sin 3s)\, ds = \frac{1}{3}\int_0^t f(s)\sin 3(t-s)\, ds\,. \end{aligned}$$

Therefore, the general solution to (2) is

$$y = C_1 \cos 3t + C_2 \sin 3t + \frac{1}{3}\int_0^t f(s)\sin 3(t-s)\, ds\,.$$

2. Note that we can rewrite (3) as

$$y'' + 9y = \cos(3t) + 3\cos t\,.$$

Therefore, the convolution formula provides that the general solution to the ODE above is

$$\begin{aligned} y &= C_1 \cos 3t + C_2 \sin 3t + \frac{1}{3}\int_0^t (\cos 3s + 3\cos s)\sin 3(t-s)\, ds \\ &= C_1 \cos 3t + C_2 \sin 3t + \frac{1}{3}\int_0^t \Big[\frac{\sin 3t + \sin(3t-6s)}{2} + 3\frac{\sin(3t-2s) + \sin(3t-4s)}{2}\Big]\, ds \\ &= C_1 \cos 3t + C_2 \sin 3t + \frac{1}{6}\Big[s \sin 3t + \frac{1}{6}\cos(3t-6s) + \frac{3}{2}\cos(3t-2s) + \frac{3}{4}\cos(3t-4s)\Big]\Big|_{s=0}^{s=t} \\ &= C_1 \cos 3t + C_2 \sin 3t + \frac{1}{6}\Big[t \sin 3t + \frac{3}{2}\cos t + \frac{3}{4}\cos t - \frac{3}{2}\cos 3t - \frac{3}{4}\cos 3t\Big] \\ &= \Big(C_1 - \frac{3}{8}\Big)\cos 3t + C_2 \sin 3t + \frac{1}{6} t \sin 3t + \frac{3}{8}\cos t\,. \end{aligned}$$

To satisfy the initial condition $y(0) = 0$, we must have $C_1 = 0$. To satisfy the initial condition $y'(0) = 0$, we must have $C_2 = 0$. Therefore, the solution is $y = \frac{1}{6} t \sin 3t + \frac{3}{8}(\cos t - \cos 3t)$.

Problem 3. Solve the differential equation

$$\frac{\sin^2(2x)}{4}y''(x) + \sin(2x)\cos^2 x y'(x) - 2y(x) = 0\,, \qquad 0 < x < \frac{\pi}{2} \tag{0.1}$$

following the steps below:

(1) (10%) Let $t = \tan x$ and $z(t) = y(\arctan t)$. Find the corresponding differential equation that z satisfies (the function arctan is identical to $\tan^{-1}$).

(2) (15%) Find the general solution to the equation for z, and then use it to find a solution to (0.1).

Solution:

(1) Let $t = \tan x$ and $z(t) = y(\tan^{-1} t)$. Then

$$z'(t) = y'(\tan^{-1} t)\frac{1}{1+t^2} \quad \text{and} \quad z''(t) = y''(\tan^{-1} t)\frac{1}{(1+t^2)^2} + y'(\tan^{-1} t)\frac{-2t}{(1+t^2)^2}\,.$$

Therefore,

$$y'(\tan^{-1} t) = (1+t^2)z'(t) \quad \text{and} \quad y''(\tan^{-1} t) = (1+t^2)^2 z''(t) + 2t(1+t^2)z'(t)\,.$$

Letting $x = \tan^{-1} t$ in the ODE we find that

$$\frac{t^2}{(1+t^2)^2}\,y''(\tan^{-1} t) + \frac{2t}{(1+t^2)^2}y'(\tan^{-1} t) - 2y(\tan^{-1} t) = 0\,;$$

thus

$$t^2 z''(t) + 2tz'(t) - 2z(t) = 0\,.$$

(2) Let r satisfy $r(r-1) + 2r - 2 = 0$. Then $r^2 + r - 2 = 0$ which implies $r = -2$ and $r = 1$. Therefore, the general solution of (0.1) is

$$z(t) = C_1 t^{-2} + C_2 t\,.$$

Therefore,

$$y(x) = z(\tan x) = C_1 \cot^2 x + C_2 \tan x\,.$$

Problem 4. (1) (15%) Let $\{\varphi_1, \varphi_2, \cdots, \varphi_{n-1}, \varphi_n\}$ be a linear independent set of n-times continuously differentiable functions on an interval $(a, b) \subseteq \mathbb{R}$. Show that there exists a set of continuous functions $\{p_{n-1}, \cdots, p_1, p_0\}$ such that

$$\varphi_i^{(n)} + p_{n-1}(t)\varphi_i^{(n-1)} + \cdots + p_1(t)\varphi_i' + p_0(t)\varphi_i = 0\,.$$

(2) (10%) Find a second order linear ODE that has $\{e^t, \sin t\}$ as a fundamental set.

Proof. (1) First, we note that since $\{\varphi_1, \varphi_2, \cdots, \varphi_{n-1}, \varphi_n\}$ is a linear independent set of n-times continuously differentiable functions on an interval $(a, b) \subseteq \mathbb{R}$, the Wronskian $W(\varphi_1, \cdots, \varphi_n)(t) \neq 0$ for all $t \in (a, b)$; thus the matrix

$$\begin{bmatrix} \varphi_1 & \varphi_1' & \cdots & \varphi_1^{(n-1)} \\ \varphi_2 & \varphi_2' & \cdots & \varphi_2^{(n-1)} \\ \vdots & & \ddots & \vdots \\ \varphi_n & \varphi_n' & \cdots & \varphi_n^{(n-1)} \end{bmatrix}$$

is invertible. Let $\{p_{n-1}, p_{n-2}, \cdots, p_1, p_0\}$ be the solution to the linear system

$$\begin{bmatrix} \varphi_1 & \varphi_1' & \cdots & \varphi_1^{(n-1)} \\ \varphi_2 & \varphi_2' & \cdots & \varphi_2^{(n-1)} \\ \vdots & & \ddots & \vdots \\ \varphi_n & \varphi_n' & \cdots & \varphi_n^{(n-1)} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix} = \begin{bmatrix} -\varphi_1^{(n)} \\ -\varphi_2^{(n)} \\ \vdots \\ -\varphi_n^{(n)} \end{bmatrix}.$$

Write the equation above as $A(t)p(t) = b(t)$. Since $\varphi_1, \varphi_2, \cdots, \varphi_{n-1}, \varphi_n$ are n-times continuously differentiable functions, the determinant of A, the adjoint matrix of A, and the vector b are continuous; thus the inverse of A is a continuous matrix and the solution p is also continuous.

(2) As indicated in (1), we let p_0 and p_1 be the solution to the linear system

$$\begin{bmatrix} e^t & e^t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} -e^t \\ \sin t \end{bmatrix},$$

and we have

$$\begin{bmatrix} -e^t \\ \sin t \end{bmatrix} = \frac{1}{e^t(\cos t - \sin t)} \begin{bmatrix} \cos t & -e^t \\ -\sin t & e^t \end{bmatrix} \begin{bmatrix} -e^t \\ \sin t \end{bmatrix} = \frac{1}{(\cos t - \sin t)} \begin{bmatrix} -(\sin t + \cos t) \\ 2\sin t \end{bmatrix}.$$

Therefore, the second order ODE that has $\{e^t, \sin t\}$ as a fundamental set is

$$(\cos t - \sin t)y'' + 2\sin t y' - (\sin t + \cos t)y = 0\,.$$

□

Problem 5. (25%) Find the general solution to the ODE

$$(t^2-2t+2)y''' - t^2y'' + 2ty' - 2y = 0$$

given two of the solutions $y = \varphi_1(t) = e^t$ and $y = \varphi_2(t) = t$.

Solution: Suppose that $y = v(t)e^t$ is a solution to the ODE above. Then

$$(t^2-2t+2)(v'''+3v''+3v'+v)e^t - t^2(v''+2v'+v)e^t + 2t(v'+v)e^t - 2ve^t = 0\,;$$

thus v satisfies

$$(t^2-2t+2)v''' + \big[3(t^2-2t+2)-t^2\big]v'' + \big[3(t^2-2t+2)-2t^2+2t\big]v' = 0$$

or equivalently,

$$(t^2-2t+2)v''' + (2t^2-6t+6)v'' + (t^2-4t+6)v' = 0\,. \tag{0.2}$$

Since $y = \varphi_2(t) = t$ is also a solution of the original ODE, we find that $v = v_1(t) = te^{-t}$ is also a solution to (0.2). Therefore, by assuming that the solution v to (0.2) can be written as $v = te^{-t}u$, we find that u satisfies that

$$(t^2-2t+2)(te^{-t}u)''' + (2t^2-6t+6)(te^{-t}u)'' + (t^2-4t+6)(te^{-t}u)' = 0\,.$$

As a consequence, u satisfies

$$t(t^2-2t+2)u''' + \big[3(1-t)(t^2-2t+2) + t(2t^2-6t+6)\big]u'' = 0$$

or equivalently,

$$t(t^2-2t+2)u''' + (-t^3+3t^2-6t+6)u'' = 0\,.$$

Since

$$\frac{-t^3+3t^2-6t+6}{t(t^2-2t+2)} = -1 + \frac{t^2-4t+6}{t(t^2-2t+2)} = -1 + \frac{3}{t} + \frac{-2t+2}{t^2-2t+2}\,,$$

the integrating factor is $\exp\Big(-t + 3\log t - \log(t^2-2t+2)\Big)$; thus solving for u'' we find that

$$\Big(e^{-t}\frac{t^3}{t^2-2t+2}u''\Big)' = 0\,.$$

We note that $u(t) = \dfrac{e^t}{t}$ must be a solution to the ODE above, and $\Big(\dfrac{e^t}{t}\Big)'' = e^t\dfrac{t^2-2t+2}{t^3}$; thus

$$u'' = C_1e^t\frac{t^2-2t+2}{t^3} = C_1\Big(\frac{e^t}{t}\Big)''\,.$$

Therefore,

$$u(t) = C_1\frac{e^t}{t} + C_2t + C_3$$

which implies that the general solution to the original ODE is

$$y(t) = C_1e^t + C_2t^2 + C_3t\,.$$