Differential Equations MA2041-A Midterm Exam 1

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1	2(1)	2(2)	3(1)	3(2)	4	5	6	7	8	Total

Problem 1. (15%) Solve the differential equation $\dfrac{dy}{dt} + y = t\sin t$ with initial condition $y(0) = \dfrac{3}{2}$.

Solution: Multiplying both side of the ODE by the integrating factor e^t, we find that

$$\big(e^t y\big)' = te^t \sin t\,. \tag{0.1}$$

We need to find the anti-derivative of $te^t \sin t$ in order to solve the ODE. First we find the anti-derivative of $e^t \sin t$. Integrating by parts,

$$\begin{aligned}\int e^t \sin t\,dt &= \int \sin t d(e^t)\,dt = e^t \sin t - \int e^t \cos t\,dt = e^t \sin t - \int \cos t d(e^t)\\ &= e^t \sin t - \Big[e^t \cos t + \int e^t \sin t\,dt\Big] = e^t(\sin t - \cos t) - \int e^t \sin t\,dt\,;\end{aligned}$$

thus $\displaystyle\int e^t \sin t\,dt = \frac{1}{2}e^t(\sin t - \cos t)$. Similarly,

$$\begin{aligned}\int e^t \cos t\,dt &= \int \cos t\,d(e^t) = e^t \cos t + \int e^t \sin t\,dt\\ &= e^t \cos t + \frac{1}{2}e^t(\sin t - \cos t) = \frac{1}{2}e^t(\sin t + \cos t)\,.\end{aligned}$$

As a consequence,

$$\begin{aligned}\int te^t \sin t\,dt &= \int t\,d\Big(\frac{1}{2}e^t(\sin t - \cos t)\Big) = \frac{t}{2}e^t(\sin t - \cos t) - \frac{1}{2}\int e^t(\sin t - \cos t)\,dt\\ &= \frac{t}{2}e^t(\sin t - \cos t) - \frac{1}{4}e^t(\sin t - \cos t) + \frac{1}{4}e^t(\sin t + \cos t)\\ &= \frac{t}{2}e^t(\sin t - \cos t) + \frac{1}{2}e^t \cos t\,,\end{aligned}$$

and (0.1) implies that

$$e^t y = \frac{t}{2}e^t(\sin t - \cos t) + \frac{1}{2}e^t \cos t + C\,.$$

Therefore, $y(t) = \dfrac{t}{2}(\sin t - \cos t) + \dfrac{1}{2}\cos t + Ce^{-t}$. Using the initial data, we find that $C = 1$; thus the solution to the ODE we are interested in is

$$y(t) = \frac{t}{2}(\sin t - \cos t) + \frac{1}{2}\cos t + e^{-t}\,.$$

□

Problem 2. 1. (5%) Consider a first order homogeneous equation $\frac{dy}{dx} = G\big(\frac{y}{x}\big)$. Show that by defining $v = \frac{y}{x}$, v satisfies the ordinary differential equation $x\frac{dv}{dx} = G(v) - v$.

2. (10%) Solve the ordinary differential equation $\big(y + x\sec\frac{y}{x}\big)dx - xdy = 0$ with initial condition $y(1) = \frac{\pi}{6}$.

Solution:

1. Since $v = \frac{y}{x}$, $y = xv$; thus $\frac{dy}{dx} = v + x\frac{dv}{dx}$ which implies that $x\frac{dv}{dx} = G(v) - v$.

2. Rearranging terms, we find that
$$\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}\,.$$
Letting $v = \frac{y}{x}$, then 1 implies that
$$x\frac{dv}{dx} = \sec v + v - v = \sec v\,;$$
thus $\cos v dv = \frac{dx}{x}$. As a consequence
$$\sin v = \log|x| + C\,.$$
Since $y(1) = 1$, $v(1) = y(1)/1 = \frac{\pi}{6}$; thus $C = \sin\frac{\pi}{6} = \frac{1}{2}$. Finally,
$$y(x) = xv(x) = x\arcsin\Big(\frac{1}{2} + \log|x|\Big)\,.$$
□

Problem 3. 1. (10%) Let $M, N : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions. Suppose that

$$\frac{N_x(x,y) - M_y(x,y)}{xM(x,y) - yN(x,y)} = h(xy)$$

for some continuous function $h : \mathbb{R} \to \mathbb{R}$. Show that the ordinary differential equation $Mdx + Ndy = 0$ has an integrating factor of the form $\mu(x,y) = z(xy)$. Give the general formula for z.

2. (10%) Solve $(3y + 2xy^2)dx + (x + 2x^2y)dy = 0$ with initial data $y(1) = 1$.

Solution:

1. Consider an integrating factor of the form $\mu(x,y) = g(xy)$. Then

$$(\mu M)_y - (\mu N)_x = 0 \Rightarrow \mu(M_y - N_x) + \mu_y M - \mu_x N = 0\,.$$

Since $\mu_y(x,y) = g'(xy)x$ and $\mu_x(x,y) = g'(xy)y$, we conclude that

$$g(M_y - N_x) + g'(xM - yN) = 0\,.$$

Therefore, $g' - hg = 0$. Let H be an anti-derivative of h, then $(e^{-H}g)' = 0$ which implies that $g = e^H$ can be an integrating factor.

2. Let $M(x,y) = 3y + 2xy^2$ and $N(x,y) = x + 2x^2y$. Then

$$\frac{N_x - M_y}{xM - yN} = \frac{1 + 4xy - (3 + 4xy)}{3xy + 2x^2y^2 - (xy + 2x^2y^2)} = \frac{-1}{xy}\,.$$

Let $h(z) = \dfrac{-1}{z}$. Then $\dfrac{N_x(x,y) - M_y(x,y)}{xM(x,y) - yN(x,y)} = h(xy)$; thus 1 implies that $g(xy) = e^{-\log|xy|}$ is a valid integrating factor. As a consequence, we instead consider

$$\frac{3y + 2xy^2}{xy}dx + \frac{x + 2x^2y}{xy}dy = 0$$

or

$$\Big(\frac{3}{x} + 2y\Big)dx + \Big(\frac{1}{y} + 2x\Big)dy = 0\,.$$

The ODE above is exact; thus there exists Φ such that $\Phi_x(x,y) = \dfrac{3}{x} + 2y$ and $\Phi_y(x,y) = \dfrac{1}{y} + 2x$. Such Φ has the form

$$\Phi(x,y) = 3\log x + 2xy + \log y\,.$$

Since $y(1) = 1$, $\Phi(x,y) = 2$ is the integral curve we are looking for.

Problem 4. Suppose that the population y of a certain creature in a given area is described by the equation

$$\frac{dy}{dt} = -ay^2 + by - c\,, \tag{1}$$

where a, b, c are positive constants.

1. (5%) Provide the condition the there are two positive equilibriums solutions to (1).
2. (10%) Under condition provided in 1, suppose that the two equilibrium solution is $y = p_1$ and $y = p_2$ with $p_1 < p_2$. Show that $y(t) = p_2$ (analytically) is asymptotically unstable equilibrium solution to (1).

Solution:

1. To have two equilibrium solutions, the equation $-a\lambda^2 + b\lambda - c = 0$ must have two distinct real roots. Therefore, $b^2 - 4ac > 0$. Moreover, the smaller root must be postive; thus

   $$p_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a} > 0\,.$$

   Since $a, b, c > 0$, the inequality above holds automatically. Therefore, the only requirement for having two equilibrium solutions is $b^2 - 4ac > 0$.

2. Let $p_2 = \dfrac{b + \sqrt{b^2 - 4ac}}{2a}$. Then

   $$\begin{aligned} \frac{dy}{dt} = -ay^2 + by - c \Rightarrow \frac{dy}{(y - p_1)(y - p_2)} &= -adt \Rightarrow \Big(\frac{1}{y - p_2} - \frac{1}{y - p_1}\Big)dy = a(p_1 - p_2)dt \\ \Rightarrow \log\left|\frac{y - p_2}{y - p_1}\right| &= a(p_1 - p_2)t + C_1 \\ \Rightarrow \left|\frac{y(t) - p_2}{y(t) - p_1}\right| &= C_2 e^{a(p_1 - p_2)t}\,. \end{aligned}$$

   Since $\lim\limits_{t\to\infty} e^{a(p_1-p_2)t} = 0$, we must have $\lim\limits_{t\to\infty} y(t) = p_2$. □

Problem 5. (15%) To solve a first order equation $y' = f(t, y)$ with initial condition $y(t_0) = y_0$, one can use the improved Euler method which is the iteration method given by the

$$u_{n+1} = u_n + \frac{h}{2}\Big[f(t_n, u_n) + f\big(t_{n+1}, u_n + hf(t_n, u_n)\big)\Big]\,, \qquad u_0 = y_0$$

where with h denoting the time step, $t_n = t_0 + nh$. Use the improved Euler method to solve $y' = y$ with $y(0) = y_0$ and show that for each fixed $T = Nh$, one has $u_N \to y_0 e^T$ as $h \to 0$.

Proof. Let $T > 0$ be given, and $N = T/h$. Since $f(y) = y$, using the improved Euler we have

$$u_{n+1} = u_n + \frac{h}{2}\big(u_n + u_n + hu_n\big) = \Big(1 + h + \frac{h^2}{2}\Big)u_n\,.$$

As a consequence,

$$u_n = \Big(1 + h + \frac{h^2}{2}\Big)^n u_0 = \Big(1 + h + \frac{h^2}{2}\Big)^n y_0\,;$$

thus $u_N = \Big(1 + h + \frac{h^2}{2}\Big)^{\frac{T}{h}} y_0$. Since

$$\lim_{h\to 0}\Big(1 + h + \frac{h^2}{2}\Big)^{\frac{T}{h}} = \lim_{h\to 0}\Big(1 + h + \frac{h^2}{2}\Big)^{\frac{T}{h+h^2/2}(1+h/2)} = e^T\,,$$

we conclude that $u_N = y_0 e^T$. □

Problem 6. (15%) Let $p : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Use the Picard iteration to solve the ordinary differential equation

$$\frac{dy}{dt} + p'y = 2p'$$

with initial condition $y(0) = y_0$.

Solution: The Picard iteration is

$$\varphi_{n+1}(t) = y_0 + \int_0^t \big(2p'(s) - p'(s)\varphi_n(s)\big)\, ds = y_0 + 2\big(p(t) - p(0)\big) - \int_0^t p'(s)\varphi_n(s)\, ds$$

with initial data $\varphi_0(t) = y_0$. Letting $q(t) = p(t) - p(0)$, we obtain that $p' = q'$; thus

$$\varphi_{n+1}(t) = y_0 + 2q(t) - \int_0^t q'(s)\varphi_n(s)\, ds\,.$$

Therefore,

$$\begin{aligned}
\varphi_1(t) &= y_0 + 2q(t) - \int_0^t y_0 q'(s)\, ds = y_0 + 2q(t) - y_0 q(t) = y_0 + (2 - y_0)q(t)\,,\\
\varphi_2(t) &= y_0 + 2q(t) - \int_0^t q'(s)\big[y_0 + (2-y_0)q(s)\big]\, ds\\
&= y_0 + (2-y_0)q(t) - \frac{2-y_0}{2}\int_0^t \big(q(s)^2\big)'\, ds = y_0 + (2-y_0)q(t) - \frac{2-y_0}{2}q(t)^2\,,\\
\varphi_3(t) &= y_0 + 2q(t) - \int_0^t q'(s)\Big[y_0 + (2-y_0)q(s) - \frac{2-y_0}{2}q(s)^2\Big]\, ds\\
&= y_0 + (2-y_0)q(t) - \int_0^t \Big[\frac{2-y_0}{2}\big(q(s)^2\big)' - \frac{2-y_0}{3!}\big(q(s)^3\big)'\Big]\, ds\\
&= y_0 + (2-y_0)q(t) - \frac{2-y_0}{2!}q(t)^2 + \frac{2-y_0}{3!}q(t)^3\,.
\end{aligned}$$

We observe φ_n for $n = 1, 2, 3$ and conjecture that

$$\begin{aligned}
\varphi_n(t) &= y_0 + (2-y_0)q(t) - \frac{2-y_0}{2!}q(t)^2 + \frac{2-y_0}{3!}q(t)^3 - \frac{2-y_0}{4!}q(t)^4 + \cdots\\
&= 2 - (2-y_0)\sum_{j=0}^{n}\frac{(-1)^j}{j!}q(t)^j\,.
\end{aligned}$$

This conjecture can be proved by induction: we have established the case $n = 1$, and suppose that the above identity holds for $n = \ell$. Then for $n = \ell + 1$,

$$\begin{aligned}
\varphi_{\ell+1}(t) &= y_0 + 2q(t) - \int_0^t q'(s)\Big[2 - (2-y_0)\sum_{j=0}^{n}\frac{(-1)^j}{j!}q(t)^j\Big]\, ds\\
&= y_0 + (2-y_0)\int_0^t q'(s)\sum_{j=0}^{n}\frac{(-1)^j}{j!}q(t)^j\, ds\\
&= y_0 + (2-y_0)\sum_{j=0}^{n}\frac{(-1)^j}{(j+1)!}q(t)^{j+1} = y_0 - (2-y_0)\sum_{j=1}^{n+1}\frac{(-1)^j}{j!}q(t)^j\\
&= 2 - (2-y_0)\sum_{j=0}^{n+1}\frac{(-1)^j}{j!}q(t)^j\,.
\end{aligned}$$

Finally, we pass to the limit as $n \to \infty$ and obtain that

$$y(t) = \lim_{n\to\infty} \varphi_n(t) = 2 - (2 - y_0)\exp\big(-q(t)\big) = 2 - (2 - y_0)\exp\big(p(0) - p(t)\big)\,. \qquad \square$$

Problem 7. (10%) Let $x : \mathbb{R} \to \mathbb{R}$ be a continuous functions satisfying

$$0 \leqslant x(t) \leqslant 1 + \int_0^t (s^2+1)x(s)\,ds \qquad \forall\, t \geqslant 0\,.$$

Show that $x(t) \leqslant \exp\Big(\frac{t^3}{3} + t\Big)$ for all $t \geqslant 0$.

Proof. Let $y(t) = \int_0^t (s^2+1)x(s)\,ds$. The fundamental theorem of Calculus implies that $\frac{y'(t)}{t^2+1} = x(t)$; thus

$$y'(t) \leqslant (t^2+1) + (t^2+1)y(t)\,.$$

As a consequence,

$$\Big[\exp\Big(-\frac{t^3}{3} - t\Big)y(t)\Big]' \leqslant (t^2+1)\exp\Big(-\frac{t^3}{3} - t\Big)\,;$$

thus by the fact that $y(0) = 0$,

$$\exp\Big(-\frac{t^3}{3} - t\Big)y(t) \leqslant 1 - \exp\Big(-\frac{t^3}{3} - t\Big)\,.$$

Therefore, $y(t) \leqslant \exp\Big(\frac{t^3}{3} + t\Big) - 1$, and this further implies that

$$0 \leqslant x(t) \leqslant 1 + y(t) \leqslant \exp\Big(\frac{t^3}{3} + t\Big)\,.$$

□

Problem 8. (10%) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function, $c = f(c)$, and consider the difference equation $y_{n+1} = f(y_n)$ with y_0 given. Suppose that $\big|f'(c)\big| > 1$. Show that there exists $\delta > 0$ and $\rho > 1$ such that if $0 < |y_n - c| < \delta$, then $|y_{n+1} - c| \geqslant \rho |y_n - c|$.

Proof. By that f is twice continuously differentiable,

$$\lim_{\delta \to 0^+} \Big(|f'(c)| - \frac{\delta}{2} \max_{x \in [c-\delta, c+\delta]} |f''(x)| \Big) = |f'(c)| > 1\,;$$

thus there exists $\delta > 0$ such that $\rho(\delta) \equiv |f'(c)| - \frac{\delta}{2} \max\limits_{x \in [c-\delta, c+\delta]} |f''(x)| > 1$. Fix such $\delta > 0$ and let $\rho \equiv \rho(\delta)$. If $0 < |y_n - c| < \delta$, then Taylor's theorem implies that for some d_n in between y_n and c,

$$\begin{aligned} y_{n+1} = f(y_n) &= f(c) + f'(c)(y_n - c) + \frac{1}{2} f''(d_n)(y_n - c)^2 \\ &= c + f'(c)(y_n - c) + \frac{1}{2} f''(d_n)(y_n - c)^2 \end{aligned}$$

which further implies that

$$\begin{aligned} |y_{n+1} - c| &\geqslant |f'(c)||y_n - c| - \frac{1}{2} \max_{x \in (c-\delta, c+\delta)} |f''(x)||y_n - c|^2 \\ &= \Big(|f'(c)| - \frac{1}{2} \max_{x \in (c-\delta, c+\delta)} |f''(x)||y_n - c| \Big) |y_n - c| \\ &\geqslant \Big(|f'(c)| - \frac{1}{2} \max_{x \in (c-\delta, c+\delta)} |f''(x)|\delta \Big) |y_n - c| \geqslant \rho |y_n - c|\,. \end{aligned}$$

□