Differential Equations Recommended Exercise 4

Problem 1. Use the Picard iteration to find the solution to the ODE $y'(t) = ty(t)$ with initial value $y(0) = 3$.

Problem 2. Let $x, a : [0, T] \to \mathbb{R}$ be a non-negative continuous functions, and satisfies

$$x(t) \leqslant M + \int_0^t a(s)x(s)\, ds \qquad \forall\, t \in [0, T].$$

Show that $x(t) \leqslant M \exp\left(\int_0^t a(s)ds \right)$ for all $t \in [0, T]$.

Problem 3. Let $T > 0$ be given. Show that there exists $0 < T_0 \leqslant T$ such that if $x : [0, T] \to \mathbb{R}$ is a non-negative continuous function satisfying

$$x(t) \leqslant M + tx(t)^2 \qquad \forall\, t \in [0, T].$$

then $x(t) \leqslant 2M$ for all $t \in [0, T_0]$.