A Concise Lecture Note on Differential Equations

1 Introduction

Definition 1.1. A differential equation is a mathematical equation that relates some unknown function with its derivatives. A differential equation is called an ordinary differential equation (ODE) if it contains an unknown function of one independent variable and its derivatives. A differential equation is called a partial differential equation (PDE) if it contains unknown multi-variable functions and their partial derivatives.

Definition 1.2. A solution to a differential equation is a function that validates the differential equations.

Example 1.3. The following three differential equations are identical (with different expression):

$$\begin{aligned} y' + y &= x + 3\,, \\ \frac{dy}{dx} + y &= x + 3\,, \\ f'(x) + f(x) &= x + 3\,. \end{aligned}$$

The function $y(x) = x + 2$ (or $f(x) = x + 2$) and $y(x) = x + 2 + e^{-x}$ (or $f(x) = x + 2 + e^{-x}$) are both solutions to the differential equation above.

Example 1.4. Let $u : \left\{ \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,t) & \mapsto & u(x,t) \end{array} \right.$ be an unknown function. The differential equation

$$u_t - u_x = t - x$$

is a partial differential equation, and $u(x,t) = x^2 + xt + t^2$ is a solution to the PDE above.

Definition 1.5. The order of a differential equation is the order of the highest derivative that appears in the equation. A differential equation of order 1 is called first order, order 2 second order, etc.

Example 1.6. The differential equations in Example 1.3 and 1.4 are both first order differential equations, while the equation $y'' + xy'^3 = x^7$ and $u_t - u_{xx} = x^3 + t^5$ are second order equations.

Definition 1.7. The ordinary differential equation

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

is said to be linear if F is linear (or more precise, affine) function of the variable y, y', $\cdots$, $y^{(n)}$. A similar definition applied to partial differential equations.

1.1 Why do we need to study differential equations?

Example 1.8 (Spring with or without Friction).

$$m\ddot{x} = -kx - r\dot{x}\,.$$

Example 1.9 (Oscillating pendulum).

$$mL\ddot{\theta} = -mg\sin\theta$$

Example 1.10 (System of ODEs). Let $p : [0,\infty) \to \mathbb{R}^+$ denote the population of certain species. If there are plenty of resource for the growth of the population, the growth rate (the rate of change of the population) is proportion to the population. In other words, there exists constant $\gamma > 0$ such that

$$\frac{d}{dt}p(t) = \gamma p(t)\,.$$

The LotkaVolterra equation or the predator-prey equation:

$$\begin{aligned} p' &= \gamma p - \alpha pq\,, \\ q' &= \beta q + \delta pq\,. \end{aligned}$$

Example 1.11. A brachistochrone curve, meaning "shortest time" or curve of fastest descent, is the curve that would carry an idealized point-like body, starting at rest and moving along the curve, without friction, under constant gravity, to a given end point in the shortest time. For given two point $(0,0)$ and (a,b), where $b < 0$, what is the brachistochrone curve connecting $(0,0)$ and (a,b)?

Define

$$\mathcal{X} = \big\{h : [0,b] \to \mathbb{R} \,\big|\, h(0) = 0, h(b) = a,\ h \text{ is differentiable on (0,b)}\big\}$$

and

$$\mathcal{A} = \big\{\varphi : [0,b] \to \mathbb{R} \,\big|\, \varphi(0) = 0, \varphi(b) = 0,\ h \text{ is differentiable on (0,b)}\big\}\,,$$

and suppose that the brachistochrone curve can be expressed as $x = f(y)$ for some $f \in \mathcal{A}$. Then f the minimizer of the functional

$$T(h) = \int_0^b \frac{\sqrt{1 + h'(y)^2}}{\sqrt{-2gy}}\,dy$$

or equivalently,

$$T(f) = \min_{h\in\mathcal{X}} \int_0^b \frac{\sqrt{1 + h'(y)^2}}{\sqrt{-2gy}}\,dy\,.$$

If $\varphi : [0,b] \to \mathbb{R}$ is differentiable such that $\varphi(0) = \varphi(b) = 0$. Then for t in a neighborhood of 0, $f + t\varphi \in \mathcal{X}$; thus

$$F(t) \equiv \int_0^b \frac{\sqrt{1 + (f + t\varphi)'(y)^2}}{\sqrt{-2gy}}\,dy$$

attains its minimum at $t = 0$. Therefore,

$$F'(0) = \frac{d}{dt}\Big|_{t=0} \int_0^b \frac{\sqrt{1 + (f + t\varphi)'(y)^2}}{\sqrt{-2gy}}\,dy = 0 \qquad \forall\,\varphi \in \mathcal{A}\,.$$

By the chain rule,

$$\int_0^b \frac{f'(y)\varphi'(y)}{\sqrt{-2gy}\sqrt{1+f'(y)^2}}\,dy = 0 \qquad \forall\,\varphi \in \mathcal{A}\,.$$

Suppose in addition that f is twice differentiable, then integration-by-parts implies that

$$-\int_0^b \Big[\frac{f'(y)}{\sqrt{-2gy}\sqrt{1+f'(y)^2}}\Big]'\varphi(y)\,dy = 0 \qquad \forall\,\varphi \in \mathcal{A}$$

which further implies that

$$\Big[\frac{f'(y)}{\sqrt{-2gy}\sqrt{1+f'(y)^2}}\Big]' = 0$$

since $\varphi \in \mathcal{A}$ is chosen arbitrarily.

Question: What if we assume that $y = f(x)$ to start with? What equation must f satisfy?

Example 1.12 (Euler-Lagrange equation)**.** In general, we often encounter problems of the type

$$\min_{y\in\mathcal{A}} \int_0^a L(y, y', t)\,dt\,, \text{ where } \mathcal{A} = \big\{y : [0,a] \to \mathbb{R} \,\big|\, y(0) = y(a) = 0\big\}\,.$$

Write $L = L(p, q, t)$. Then the minimizer $y \in \mathcal{A}$ satisfies

$$\frac{d}{dt}L_q(y, y', t) = L_p(y, y', t)\,.$$

The equation above is called the Euler-Lagrange equation.

Example 1.13 (Heat equations)**.** Let $u(x,t)$ defined on $\Omega \times (0,T]$ be the temperature of a material body at point $x \in \Omega$ at time $t \in (0,T]$, and $c(x)$, $\varrho(x)$, $k(x)$ be the specific heat, density, and the inner thermal conductivity of the material body at x. Then by the conservation of heat, for any open set $\mathcal{U} \subseteq \Omega$,

$$\frac{d}{dt}\int_{\mathcal{U}} c(x)\varrho(x)u(x,t)\,dx = \int_{\partial\mathcal{U}} k(x)\nabla u(x,t)\cdot \mathrm{N}(x)\,dS\,, \tag{1.1}$$

where N denotes the outward-pointing unit normal of $\mathcal{U}$. Assume that u is smooth, and $\mathcal{U}$ is a Lipschitz domain. By the divergence theorem, (1.1) implies

$$\int_{\mathcal{U}} c(x)\varrho(x)u_t(x,t)dx = \int_{\mathcal{U}} \mathrm{div}\big(k(x)\nabla u(x,t)\big)\,dx\,.$$

Since $\mathcal{U}$ is arbitrary, the equation above implies

$$c(x)\varrho(x)u_t(x,t) - \mathrm{div}(k(x)\nabla u(x,t)) = 0 \quad \forall\, x \in \Omega\,, t \in (0,T].$$

If k is constant, then

$$\frac{c\varrho}{k}u_t = \Delta u \equiv \sum_{i=1}^{\mathrm{n}} \frac{\partial^2 u}{\partial x_i^2}\,.$$

If furthermore c and ϱ are constants, then after rescaling of time we have

$$u_t = \Delta u\,. \tag{1.2}$$

This is the standard **heat equation**, the prototype equation of **parabolic** equations.

Example 1.14 (Minimal surfaces)**.** Let Γ be a closed curve in $\mathbb{R}^3$. We would like to find a surface which has minimal surface area while at the same time it has boundary Γ.

Suppose that $\Omega \subseteq \mathbb{R}^2$ is a bounded set with boundary parametrized by $(x(t), y(t))$ for $t \in I$, and Γ is a closed curve parametrized by $(x(t), y(t), f(x(t), y(t)))$. We want to find a surface having C as its boundary with minimal surface area. Then the goal is to find a function u with the property that $u = f$ on $\partial\Omega$ that minimizes the functional

$$\mathscr{A}(w) = \int_\Omega \sqrt{1 + |\nabla w|^2}\, dA\,.$$

Let $\varphi \in \mathscr{C}^1(\overline{\Omega})$, and define

$$\delta\mathcal{A}(u; \varphi) = \lim_{t\to 0} \frac{\mathcal{A}(u + t\varphi) - \mathcal{A}(u)}{t} = \int_\Omega \frac{\nabla u \cdot \nabla\varphi}{\sqrt{1 + |\nabla u|^2}}\, dx\,.$$

If u minimize $\mathcal{A}$, then $\delta\mathcal{A}(u; \varphi) = 0$ for all $\varphi \in \mathscr{C}_c^1(\Omega)$. Assuming that $u \in \mathscr{C}^2(\overline{\Omega})$, we find that u satisfies

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) = 0\,,$$

or expanding the bracket using the Leibnitz rule, we obtain the *minimal surface equation*

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0 \quad \forall\, (x, y) \in \Omega\,. \tag{1.3}$$

Example 1.15 (System of PDEs - the Euler equations)**.** Let $\Omega \subseteq \mathbb{R}^3$ denote a fluid container, and $\varrho(x, t), \boldsymbol{u}(x, t), p(x, t)$ denotes the fluid density, velocity and pressure at position x and time t. For a given an open subset $\mathcal{O} \subseteq \Omega$ with smooth boundary, the rate of change of the mass in $\mathcal{O}$ is the same as the mass flux through the boundary; thus

$$\frac{d}{dt}\int_{\mathcal{O}} \varrho(x, t)dx = -\int_{\partial\mathcal{O}} (\varrho\boldsymbol{u})(x, t) \cdot \mathbf{N}\, dS\,,$$

where $\mathbf{N}$ is the outward-pointing unit normal of ∂O. The divergence theorem then implies that

$$\frac{d}{dt}\int_{\mathcal{O}} \varrho(x, t)dx = -\int_{\mathcal{O}} \operatorname{div}(\varrho\boldsymbol{u})(x, t)\, dS\,.$$

If ϱ is a smooth function, then $\frac{d}{dt}\int_{\mathcal{O}} \varrho(x, t)dx = \int_{\mathcal{O}} \varrho_t(x, t)dx$; thus

$$\int_{\mathcal{O}} \big[\varrho_t + \operatorname{div}(\varrho\boldsymbol{u})\big](x, t)dx = 0\,.$$

Since $\mathcal{O}$ is chosen arbitrarily, we must have

$$\varrho_t + \operatorname{div}(\varrho\boldsymbol{u}) = 0 \qquad \text{in} \quad \Omega\,. \tag{1.4}$$

Equation (1.4) is called the equation of continuity.

Now we consider that conservation of momentum. Let $\boldsymbol{m} = \varrho\boldsymbol{u}$ be the momentum. The conservation of momentum states that

$$\frac{d}{dt}\int_{\mathcal{O}} \boldsymbol{m}\, dx = -\int_{\partial\mathcal{O}} \boldsymbol{m}(\boldsymbol{u} \cdot \mathbf{N})\, dS - \int_{\partial\mathcal{O}} p\mathbf{N}\, dS + \int_{\mathcal{O}} \varrho\boldsymbol{f}\, dx\,,$$

here we use the fact that the rate of change of momentum of a body is equal to the resultant force acting on the body, and with p denoting the pressure the buoyancy force is given by $\displaystyle\int_{\partial\mathcal{O}} p\mathbf{N}\,dS$. Here we assume that the fluid is *invicid* so that no friction force is presented in the fluid. Therefore, assuming the smoothness of the variables, the divergence theorem implies that

$$\int_{\mathcal{O}}\Big[\boldsymbol{m}_t+\sum_{j=1}^{\mathrm{n}}\frac{\partial(\boldsymbol{m}\boldsymbol{u}^j)}{\partial x_j}+\nabla p-\varrho\boldsymbol{f}\Big]dx=0\,.$$

Since $\mathcal{O}$ is chosen arbitrarity, we obtain the momentum equation

$$(\varrho\boldsymbol{u})_t+\mathrm{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u})=-\nabla p+\varrho\boldsymbol{f}\,. \tag{1.5}$$

Initial conditions: $\varrho(x,0)=\varrho_0(x)$ and $\boldsymbol{u}(x,0)=\boldsymbol{u}_0(x)$ for all $x\in\Omega$.
Boundary condition: $\boldsymbol{u}\cdot\mathbf{N}=0$ on $\partial\Omega$.

1. If the density is constant (such as water), then (1.4) and (1.5) reduce to

$$\boldsymbol{u}_t+\boldsymbol{u}\cdot\nabla\boldsymbol{u}=-\nabla p+\boldsymbol{f}\qquad\text{in}\quad\Omega\times(0,T)\,, \tag{1.6a}$$

$$\mathrm{div}\boldsymbol{u}=0\qquad\text{in}\quad\Omega\times(0,T)\,. \tag{1.6b}$$

   Equation (1.6) together with the initial and the boundary condition are called the *incompressible Euler equations*.

2. If the pressure p solely depends on the density; that is, $p=p(\varrho)$ (the equation of state), then (1.4) and (1.5) together with are called the *isentropic Euler equations*.

1.2 Direction Fields

A direction field is in particular very useful in the study of first order differential equations of the type:

$$\frac{dy}{dt}=f(t,y)\,,$$

where f is a scalar function. A direction field is a vector-field on the (t,y)-plane on which a vector $(1,f(t,y))$ is associated with each point (t,y).

Example 1.16. Consider a falling object whose velocity satisfies the ODE

$$m\frac{dv}{dt}=mg-\gamma v\,.$$

1.3 Initial and Boundary Conditions

Given y satisfies $f\big(t,y,y',\cdots,y^{(n)}\big)=0$, the initial condition for the ODE is of the form

$$y(a)=b_1\,,y'(a)=b_2\,,\cdots,y^{(n-1)}(a)=b_n$$

which specify the derivative of y at a up to $(n-1)$-th derivative of y.

If we are interested in an ODE of the form $f\big(x,y,y',y'',\cdots,y^{(2n-1)},y^{(2n)}\big)=0$ on a particular interval $[a,b]$, the boundary condition for an ODE of this type is of the form

$$y(a)=c_1,y(b)=d_1,y'(a)=c_2,y'(b)=d_2,\cdots,y^{(n)}(a)=c_{n+1},y^{(n)}(b)=d_{n+1}\,.$$

2 First Order Differential Equations

In general, a first order ODE can be written as

$$\frac{dy}{dt} = f(t, y)$$

for some function f. In this chapter, we are going to solve the linear equation above explicitly with

1. $f(t, y) = p(t)y + q(t)$;

2. $f(t, y) = g(y)h(t)$,

and also provide some insight of nonlinear equations.

2.1 Linear Equations; Method of Integrating Factors

Suppose that we are given a first order linear equation

$$\frac{dy}{dt} + p(t)y = q(t) \quad \text{with initial condition } y(a) = b\,.$$

Let $P(t)$ be an anti-derivative of $p(t)$; that is, $P'(t) = p(t)$. Then

$$\begin{aligned}
&e^{P(t)}\Big(\frac{dy}{dt} + P'(t)y\Big) = e^{P(t)}q(t) \Rightarrow \frac{d}{dt}\Big(e^{P(t)}y(t)\Big) = e^{P(t)}q(t)\\
&\Rightarrow \int_a^t \frac{d}{ds}\Big(e^{P(s)}y(s)\Big)ds = \int_a^t e^{P(s)}Q(s)ds \Rightarrow e^{P(t)}y(t) - e^{P(a)}y(a) = \int_a^t e^{P(s)}Q(s)ds\\
&\Rightarrow y(t) = e^{P(a)-P(t)}b + \int_a^t e^{P(s)-P(t)}Q(s)ds\,.
\end{aligned}$$

How about if we do not know what the initial data is? Then

$$e^{P(t)}\Big(\frac{dy}{dt} + P'(t)y\Big) = e^{P(t)}q(t) \Rightarrow \frac{d}{dt}\Big(e^{P(t)}y(t)\Big) = e^{P(t)}q(t) \Rightarrow e^{P(t)}y(t) = C + \int e^{P(t)}q(t)dt\,,$$

where $\int e^{P(t)}q(t)dt$ denotes an anti-derivative of $e^P Q$. Therefore,

$$y(t) = Ce^{-P(t)} + e^{-P(t)}\int e^{P(t)}q(t)dt$$

Example 2.1. Solve $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$. Answer: $y(t) = \frac{3}{5}e^{t/3} + Ce^{-t/2}$.

Example 2.2. Solve $\frac{dy}{dt} - 2y = 4 - t$. Answer: $y(t) = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$.

Example 2.3. Solve $ty' + 2y = 4t^2$ with $y(1) = 2$. Answer: $y(t) = t^2 + \frac{1}{t^2}$.

2.2 Separable Equations

Suppose that we are given a first order linear equation

$$\frac{dy}{dt} = g(y)h(t) \quad \text{with initial condition } y(a) = b\,,$$

where $1/g$ is assume to be integrable. Let G be an anti-derivative of $1/g$. Then

$$\begin{aligned}
&\frac{dy}{dt} = g(y)h(t) \Rightarrow \frac{1}{g(y)}\frac{dy}{dt} = h(t) \Rightarrow G'(y)\frac{dy}{dt} = h(t) \\
&\Rightarrow \frac{d}{dt}G(y(t)) = h(t) \Rightarrow \int_a^t \frac{d}{ds}G(y(s))ds = h(t) \Rightarrow G(y(t)) - G(y(a)) = \int_a^t h(s)ds \\
&\Rightarrow G(y(t)) = G(b) + \int_a^t h(s)ds\,,
\end{aligned}$$

and y can be solved if the inverse function of G is known.

Example 2.4. Let y be a solution to the ODE $\dfrac{dy}{dx} = \dfrac{x^2}{1-y^2}$. Then x, y satisfies $x^3 + y^3 - 3y = C$ for some constant C.

Example 2.5. Let y be a solution to the ODE $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$ with initial data $y(0) = -1$. Then $y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$.

Definition 2.6 (Integral Curves)**.** Let $\mathbf{F} = (F_1, \cdots, F_n)$ be a vector field. A parametric curve $\boldsymbol{x}(t) = \big(x_1(t), \cdots, x_n(t)\big)$ is said to be an ***integral curve*** of $\mathbf{F}$ if it is a solution of the following autonomous system of ODEs:

$$\begin{aligned}
\frac{dx_1}{dt} &= F_1(x_1, \cdots, x_n)\,, \\
&\vdots \\
\frac{dx_n}{dt} &= F_n(x_1, \cdots, x_n)\,.
\end{aligned}$$

In particular, when $n = 2$, the autonomous system above is reduced to

$$\frac{dx}{dt} = F(x,y)\,, \qquad \frac{dy}{dt} = G(x,y) \tag{2.1}$$

for some function F, G. Since at each point $(x_0, y_0) = \big(x(t_0), y(t_0)\big)$ on the integral curve,

$$\frac{dy}{dx}\bigg|_{(x,y)=(x_0,y_0)} = \frac{dy/dt}{dx/dt}\bigg|_{t=t_0}$$

if $\dfrac{dx}{dt}\Big|_{t=t_0} \neq 0$, instead of finding solutions to (2.1) we often solve

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}\,.$$

Example 2.7. Find the integral curve of the vector field $\mathbf{F}(x,y) = (4 + y^3, 4x - x^3)$ passing through $(0,1)$. Answer: $y^4 + 16y + x^4 - 8x^2 = 17$.

2.3 Modelling with First Order Equations

Example 2.8 (Mixing)**.** At the very beginning, Q_0 Kgs salt were dissolved in 100 liters of water. Afterward, salty water containing 1/4 Kg salt per liter enter the container at the speed r liters per minute, while at the same time r liters of the well-mixed solution leaves the tank every minute. If $Q(t)$ is the quantity (in Kgs) of salt in the container at time t, then

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}, \qquad Q(0) = Q_0 .$$

To solve this ODE, we use the integrating factor and obtain that

$$\begin{aligned}\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4} \Rightarrow \frac{d}{dt}\big(e^{rt/100}Q(t)\big) = \frac{r}{4}e^{rt/100} \Rightarrow e^{rt/100}Q(t) = 25e^{rt/100} + C \\ \Rightarrow Q(t) = 25 + Ce^{-rt/100}\end{aligned}$$

and the initial data implies that $C = Q_0 - 25$. Therefore,

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100} .$$

Using the separation of variables,

$$\frac{dQ}{dt} = \frac{r}{100}(25 - Q) \Rightarrow \frac{dQ}{25 - Q} = \frac{r}{100}dt \Rightarrow -\log|25 - Q(t)| = \frac{rt}{100} + C$$

and the initial data implies that $C = -\log|25 - Q_0|$. Therefore,

$$\frac{|25 - Q_0|}{|25 - Q(t)|} = e^{rt/100} \quad \text{or} \quad Q(t) = 25 + (Q_0 - 25)e^{-rt/100} .$$

Example 2.9 (Escape Velocity)**.** By Newton's second law of motion $F = ma$, we consider the equation $m\frac{dv}{dt} = -\frac{GMm}{(R+x)^2}$. Note that on the surface $x = 0$, the forcing equals $-mg$; thus $\frac{GM}{R^2} = g$. In other words, the equation becomes $m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}$.

Suppose that v can be written as a function of the position x, then

$$\begin{aligned}m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2} \Rightarrow \frac{dv}{dx}\frac{dx}{dt} = -\frac{gR^2}{(R+x)^2} \Rightarrow v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2} \Rightarrow vdv = -\frac{gR^2}{(R+x)^2}dx \\ \Rightarrow \frac{1}{2}v^2 = \frac{gR^2}{R+x} + C \Rightarrow v(x) = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}} ,\end{aligned}$$

where $v_0 = v(0)$ is the initial data. For a given v_0, the maximum attitude ξ that the body reaches is given by $\xi = \frac{v_0^2 R}{2gR - v_0^2}$, and to escape the gravity of the earth, the initial velocity v_0 should be not less than $\sqrt{2gR}$.

2.4 Differences b/w Linear and Nonlinear Equations

Concerns in differential equations: ***existence*** and ***uniqueness*** of solutions to differential equations.

Theorem 2.10. *Let the function f be functions of t and y such that f and its partial derivative $\dfrac{\partial f}{\partial y}$ is continuous in some rectangular domain $(t, y) \in R \equiv (\alpha, \beta) \times (\gamma, \delta)$. Suppose that $(t_0, y_0) \in R$. Then in some interval $t \in (t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$, there exists a unique solution $y = \varphi(t)$ to the initial value problem*

$$y' = f(t, y) \qquad y(t_0) = y_0 \,.$$

Example 2.11. Consider $\dfrac{dy}{dt} = y^{1/3}$ with initial data $y(0) = 0$. There are infinitely many solutions

$$y(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < t_0 \,, \\ \pm\big[\frac{2}{3}(t - t_0)\big]^{\frac{3}{2}} & \text{if } t \geqslant t_0 \,. \end{cases}$$

The reason for non-uniqueness of the solutions is that $\dfrac{\partial f}{\partial y}$ is not continuous near $(0, 0)$.

Let us look at what separation of variables implies. Using the separation of variables, with $G(y) = \dfrac{3}{2} y^{3/2}$ we have

$$\frac{dy}{dt} = y^{1/3} \;\Rightarrow\; y^{-1/3}\frac{dy}{dt} = 1 \;\Rightarrow\; G'(y)\frac{dy}{dt} = 1 \Rightarrow \frac{dG}{dt} = 1 \,.$$

We cannot apply the fundamental theorem to conclude that $G(t) = t + C$ here since $\dfrac{dG}{dt}$ is not continuous in the time interval containing $t = 0$ (in fact, $\displaystyle\int \frac{dG}{dt} dt$ is an improper integral). However, if we apply the fundamental theorem of calculus, we obtain that

$$G(y(t)) = t + C \;\Rightarrow\; y(t) = \big[\frac{2}{3}t\big]^{\frac{3}{2}}$$

which is one of the solutions.

2.5 Autonomous Equations and Population Dynamics

Definition 2.12. A first order ODE $f(t, y, y') = 0$ is called ***autonomous*** if it can be rewritten as

$$\frac{dy}{dt} = f(y) \,.$$

Example 2.13 (Exponential Growth)**.** In Chapter 1 we have discussed the equation

$$\frac{dp}{dt} = \gamma p \,,$$

where p is the population of certain species and γ is the rate of growth (or decline). Solving the ODE with the initial data $p(0) = p_0$, we obtain that

$$p(t) = p_0 e^{\gamma t} \,.$$

Example 2.14 (Logistic Growth)**.** Instead of the purely theoretical model in Example 2.13, we consider the equation

$$\frac{dp}{dt} = h(p)p \,,$$

where the growth rate depends on the population. The simplest function for h is $h(p) = \gamma - \alpha p$ for some positive constant α. Then

$$\frac{dp}{dt} = (\gamma - \alpha p)p \qquad \text{or equivalently} \qquad \frac{dp}{dt} = \gamma\Big(1 - \frac{p}{K}\Big)p \tag{2.2}$$

in which $K = \frac{\gamma}{\alpha}$. Equation (2.2) is called the ***logistic*** equation.

Equilibrium solution: An equilibrium solution to a differential equation is a solution which does not vary with its independent variable (usually time). Therefore, there are two equilibrium solutions to (2.2): $p = \varphi_1(t) = 0$ and $p = \varphi_2(t) = K$.

General solution: Let $p_0 = p(0) > 0$ be the initial data. If $p_0 \neq 0$ or K, using separation of variables:

$$\frac{K dp}{(K-p)p} = \gamma dt \;\Rightarrow\; \Big(\frac{1}{K-p} + \frac{1}{p}\Big)dp = \gamma dt \;\Rightarrow\; -\log|K-p| + \log|p| = \gamma t + C$$
$$\Rightarrow\; \frac{p}{|K-p|} = \frac{p_0}{|K-p_0|}e^{\gamma t}\,.$$

Therefore, $p(t) = \dfrac{Kp_0}{p_0 + (K-p_0)e^{-\gamma t}}$ which implies that $p \to K$ as $t \to \infty$, no matter $p_0 > K$ or $0 < p_0 < K$. The solution $p = \varphi_2(t) = K$ is then called an ***asymptotically stable solution***, while $p = \varphi_1(t) = 0$ is an ***unstable equilibrium solution***. The number K is called the ***saturation level*** or the ***environmental carrying capacity***.

Note that since

$$\frac{d^2p}{dt^2} = \frac{d}{dt}\frac{dp}{dt} = \frac{d}{dt}f(p) = f'(p)\frac{dp}{dt} = f'(p)f(p)\,,$$

the graph of p versus t is concave up when f and f' have the same sign, while the graph is concave down when f and f' have opposite signs. Therefore, solutions are concave up for $0 < y < \frac{K}{2}$ and$y > K$, while the solutions are concave down for $\frac{K}{2} < y < K$.

Example 2.15 (A Critical Threshold)**.** In Example 2.14, what happened if $\gamma < 0$? In this case, we instead consider

$$\frac{dp}{dt} = -\gamma\Big(1 - \frac{p}{T}\Big)p\,, \tag{2.3}$$

where $\gamma > 0$ and $T > 0$. This time the solution is

$$p(t) = \frac{Tp_0}{p_0 + (T-p_0)e^{\gamma t}}$$

Unless $p_0 \geqslant T$, the population decays to zero; thus T is called the ***threshold level*** which means below this level the growth of population does not occur. When $p_0 > T$, the time $T^* = \frac{1}{r}\log\frac{p_0}{p_0 - T}$ to which the population tends to infinite; thus the population becomes unbounded in a finite time.

The equilibrium solution $p(t) = 0$ is an ***asymptotically stable solution***, while the equilibrium solution $p(t) = T$ is an ***asymptotically unstable solution***.

Example 2.16 (Logistic Growth with a Threshold)**.** Combining the experiences from the previous two examples, we design an model which cooperates the two phenomena:

1. the population will not grow if the initial population is below certain threshold;
2. the population will not blow up in a finite time if the population will grow.

Instead of letting $h(p) = \gamma - \alpha p$, we consider the following more complicated situation: $h(p) = -\gamma\big(1 - \frac{p}{T}\big)\big(1 - \frac{p}{K}\big)$ for some $\gamma > 0$ and $0 < T < K$.
Equilibrium solution: $\varphi_1(t) = 0$, $\varphi_2(t) = T$, $\varphi_3(t) = K$. φ_1 and φ_3 are asymptotically stable, while φ_2 is asymptotically unstable.
General solution: (Important or not?)

2.6 Exact Equations and Integrating Factors

Recall vector calculus:

Definition 2.17 (Vector fields)**.** A ***vector field*** is a vector-valued function whose domain and range are subsets of Euclidean space $\mathbb{R}^n$.

Definition 2.18 (Conservative vector fields)**.** A vector field $\mathbf{F} : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be ***conservative*** if $\mathbf{F} = \nabla\varphi$ for some scalar function φ. Such a φ is called a (scalar) potential for $\mathbf{F}$ on $\mathcal{D}$.

Theorem 2.19. *If $\mathbf{F} = (M, N)$ is a conservative vector field in a domain $\mathcal{D}$, then $N_x = M_y$ in $\mathcal{D}$.*

Theorem 2.20. *Let $\mathcal{D}$ be an open, connected domain, and let $\mathbf{F}$ be a smooth vector field defined on $\mathcal{D}$. Then the following three statements are equivalent:*

1. *$\mathbf{F}$ is conservative in $\mathcal{D}$.*
2. *$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth, closed curve C in $\mathcal{D}$.*
3. *Given and two point $P_0, P_1 \in \mathcal{D}$, $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all piecewise smooth curves in D starting at P_0 and ending at P_1.*

Definition 2.21. A connected domain $\mathcal{D}$ is said to be ***simply connected*** if every simple closed curve can be continuously shrunk to a point in $\mathcal{D}$ without any part ever passing out of $\mathcal{D}$.

Theorem 2.22. *Let $\mathcal{D}$ be a simply connected domain, and M, N, M_y, N_x be continuous in $\mathcal{D}$. If $M_y = N_x$, then $\mathbf{F} = (M, N)$ is conservative.*

Sketch of the proof. Since $N_x = M_y$,

$$\begin{aligned} N(x,y) &= N(x_0,y) + \int_{x_0}^{x} M_y(z,y)\,dz = N(x_0,y) + \frac{\partial}{\partial y}\int_{x_0}^{x} M(z,y)\,dz \\ &= \frac{\partial}{\partial y}\left[\Psi(y) + \int_{x_0}^{x} M(z,y)\,dz\right], \end{aligned}$$

where $\Psi(y)$ is an anti-derivative of $N(x_0, y)$. Let $\varphi(x, y) = \Psi(y) + \int_{x_0}^{x} M(z, y)\,dz$. Then clearly $(M, N) = \nabla\varphi$ which implies that $\mathbf{F} = (M, N)$ is conservative. □

Combining Theorem 2.19 and 2.22, in a simply connected domain a vector field $\mathbf{F} = (M, M)$ is conservative if and only if $M_y = N_x$.

Example 2.23. Let $\mathcal{D} = \mathbb{R}^2 \backslash \{(0,0)\}$, and $M(x,y) = \dfrac{-y}{x^2+y^2}$, $N(x,y) = \dfrac{x}{x^2+y^2}$. Then $M_y = N_x = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ in $\mathcal{D}$; however, $F \neq \nabla\varphi$ for some scalar function φ for it there exists such a φ, φ, up to adding a constant, must be identical to the polar angle $\theta(x,y) \in [0, 2\pi)$.

Now suppose that we are given a differential equation of the form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)},$$

in which separation of variables is not possible. We would like to find integral curves of the vector field $\mathbf{F} = (-N, M)$. Note that the ODE above is equivalent to that

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

Definition 2.24. An ODE of the form $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$ is called ***exact*** if there exists a continuously differentiable function φ, called the potential function, such that $\varphi_x = M$ and $\varphi_y = N$.

To solve the ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \tag{2.4}$$

the following two possibilities are most possible situations:

1. If $M_y = N_x$ in a simply connected domain $\mathcal{D}$, then Theorem 2.22 implies that the ODE (2.4) is exact in a simply connected domain $\mathcal{D} \subseteq \mathbb{R}^2$; that is, there exists a potential function φ such that $\nabla\varphi = (M, N)$. Then (2.4) can be rewritten as

   $$\varphi_x(x,y) + \varphi_y(x,y)\frac{dy}{dx} = 0;$$

   and if $(x(t), y(t))$ is an integral curve, we must have

   $$\varphi_x(x(t),y(t))\frac{dx}{dt} + \varphi_y(x(t),y(t))\frac{dy}{dt} = 0 \quad \text{or equivalently,} \quad \frac{d}{dt}\varphi(x(t),y(t)) = 0.$$

   Therefore, integral curve satisfies $\varphi(x,y) = C$.

2. If $M_y \neq N_x$, we look for a function μ such that $(\mu M)_y = (\mu N)_x$ in a simply connected domain $\mathcal{D} \subseteq \mathbb{R}^2$. Such a μ always exists (in theory, but may be hard to find the explicit expression), and such a μ is called an ***integrating factor***.

   If such a μ exists, then μ satisfies

   $$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$

   Usually solving a PDE as above is as difficult as solving the original ODE.

Example 2.25. Solve $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\dfrac{dy}{dx} = 0$.

Let $M(x,y) = y\cos x + 2xe^y$ and $N(x,y) = \sin x + x^2e^y - 1$. Then $M_y(x,y) = \cos \boldsymbol{x} + 2xe^y = N_x(x,y)$; thus the ODE above is exact. To find the potential function φ, due to the fact that $\varphi_x = M$ we find that

$$\varphi(x,y) = \Psi(y) + \int M(x,y)dx = \Psi(y) + y\sin x + x^2e^y$$

for some function Ψ. By $\varphi_y = N$, we must have $\Psi'(y) = -1$. Therefore, $\Psi(y) = -y + C$; thus the potential function φ is

$$\varphi(x,y) = y\sin x + x^2e^y - y + C\,.$$

Example 2.26. Solve $(3xy + y^2) + (x^2 + xy)\dfrac{dy}{dx} = 0$.

Let $M(x,y) = 3xy + y^2$ and $N(x,y) = x^2 + xy$. Then $M_y - N_x = x + y$. Assuming that the integrating factor μ is only a function of x, then μ satisfies

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = \frac{1}{x}\mu\,;$$

thus $\mu(x) = x$.

Multiplying both side of the ODE by μ, we then obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)\frac{dy}{dx} = 0$$

which is exact, and the integral curves of the ODE above, by finding the potential, satisfies

$$x^3y + \frac{x^2y^2}{2} = C\,.$$

One can also verify that the function $\mu(x,y) = \dfrac{1}{xy(2x+y)}$ is also a valid integrating factor.

2.7 Numerical Approximations: Euler's Method

The goal in this section is to solve the ODE

$$\frac{dy}{dt} = f(t,y) \qquad y(t_0) = y_0 \tag{2.5}$$

numerically (meaning, programming in computer to produce an approximation of the solution) in the time interval $[t_0, t_0 + T]$.

Let Δt denote the time step (which mean we only care what the approximated solution is at time $t_k = t_0 + k\Delta t$ for all $k \in \mathbb{N}$), and $y_k = y(t_0 + k\Delta t)$. Since $\dfrac{dy}{dt}(t_k) \approx \dfrac{y_{k+1} - y_k}{\Delta t}$ when $\Delta t \approx 0$, we substitute $\dfrac{y_{k+1} - y_k}{\Delta t}$ for $\dfrac{dy}{dt}(t_k)$ and obtain

$$y_{k+1} \approx y_k + f(t_k, y_k)\Delta t \qquad \forall\, k \in \mathbb{N}\,.$$

The ***forward/explicit Euler method*** is the iterative scheme

$$u_{k+1} = u_k + f(t_k, u_k)\Delta t \qquad \forall\, k \in \big\{1, 2, \cdots, [\tfrac{T}{\Delta t}] - 1\big\}\,, u_0 = y_0 \text{ (in theory)}\,. \tag{2.6}$$

Assume that f is bounded and has bounded continuous partial derivatives f_t and f_y; that is, f_t and f_y are continuous and for some constant $M > 0$ $|f(t,y)| + |f_t(t,y)| + |f_y(t,y)| \leqslant M$ for all t, y. Then the mean value theorem implies that the fundamental theorem of ODE (which will be provided in the next section) provides a unique continuously differentiable solution $y = y(t)$ to (2.5). Since f_t and f_y are continuous, we must have that y is twice continuously differentiable since

$$y'' = f_t(t,y) + f_y(t,y)y' .$$

By Taylor's theorem, for some $\theta_k \in (0,1)$ we have

$$\begin{aligned} y(t_{k+1}) &= y(t_k) + y'(t_k)\Delta t + \frac{1}{2}(\Delta t)^2 y''(t_k + \theta_k \Delta t) \\ &= y_k + f(t_k, y_k)\Delta t + \frac{(\Delta t)^2}{2}\big[f_t + f_y f\big](t_k + \theta_k\Delta t, y(t_k + \theta_k \Delta t)) ; \end{aligned}$$

thus we conclude that

$$y_{k+1} = y_k + f(t_k, y_k)\Delta t + \frac{\Delta t}{2}\tau_k$$

for some τ_k satisfying $|\tau_k| \leqslant L\Delta t$ for some constant L.

With e_k denoting $u_k - y_k$, we have

$$e_{k+1} = e_k + \big[f(t_k, u_k) - f(t_k, y_k)\big]\Delta t + \frac{\Delta t}{2}\tau_k .$$

The mean value theorem then implies that

$$|e_{k+1}| \leqslant |e_k| + (M\Delta t)|e_k| + \frac{L}{2}(\Delta t)^2 = (1 + M\Delta t)|e_k| + \frac{L}{2}(\Delta t)^2 ;$$

thus by iteration we have

$$\begin{aligned} |e_{k+1}| &\leqslant (1 + M\Delta t)|e_k| + \frac{L}{2}(\Delta t)^2 \leqslant (1 + M\Delta t)\big[(1 + M\Delta t)|e_{k-1}| + \frac{L}{2}(\Delta t)^2\big] + \frac{L}{2}(\Delta t)^2 \\ &= (1 + M\Delta t)^2|e_{k-1}| + \frac{L}{2}(\Delta t)^2\big[1 + (1 + M\Delta t)\big] \\ &\leqslant \cdots\cdots\cdots \\ &\leqslant (1 + M\Delta t)^{k+1}|e_0| + \frac{L}{2}(\Delta t)^2\big[1 + (1 + M\Delta t) + (1 + M\Delta t)^2 + \cdots + (1 + M\Delta t)^k\big] \\ &= (1 + M\Delta t)^{k+1}|e_0| + \frac{L}{2M}\Delta t\big[(1 + M\Delta t)^{k+1} - 1\big] \\ &\leqslant (1 + M\Delta t)^{k+1}\Big(|e_0| + \frac{L}{2M}\Delta t\Big) \end{aligned}$$

for all $k \in \big\{1, 2, \cdots, [\frac{T}{\Delta t}] - 1\big\}$. Since $(1 + M\Delta t) \leqslant e^{M\Delta t}$, we conclude that

$$|e_{k+1}| \leqslant e^{M(k+1)\Delta t}\Big(|e_0| + \frac{L}{2M}\Delta t\Big) \leqslant e^{MT}\Big(|e_0| + \frac{L}{2M}\Delta t\Big)$$

which further implies that

$$\max_{k \in \{1, \cdots, [\frac{T}{\Delta}]\}} |e_k| \leqslant e^{MT}\Big(|e_0| + \frac{L}{2M}\Delta t\Big) .$$

2.8 The Existence and Uniqueness Theorem

In this section we prove Theorem 2.10. Recall that

Theorem 2.10. *Let f be a function of t and y such that f and its partial derivative $\dfrac{\partial f}{\partial y}$ is continuous in some rectangular domain $(t, y) \in R \equiv (\alpha, \beta) \times (\gamma, \delta)$. Suppose that $(t_0, y_0) \in R$. Then in some interval $t \in (t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$, there exists a unique solution $y = \varphi(t)$ to the initial value problem*

$$y' = f(t, y) \qquad y(t_0) = y_0 \,. \tag{2.7}$$

Proof. The proof is separated into two parts.

Existence: Choose a constant $k \in (0, 1)$ such that $I \times J = [t_0 - k, t_0 + k] \times [y_0 - k, y_0 + k] \subseteq R$. Since $I \times J$ is closed and bounded, $|f|$ and $|f_y|$ attain their maximum in $I \times J$. Assume that for some $M \geqslant 1$, $\big|f(t, y)\big| + \big|f_y(t, y)\big| \leqslant M$ for all $(t, y) \in I \times J$. Let $h = k/M$ and $I_h = [t_0 - h, t_0 + h]$. Then for $t \in I_h$, define the iterative scheme (called ***Picard's iteration***)

$$\varphi_{n+1}(t) = y_0 + \int_{t_0}^{t} f\big(s, \varphi_n(s)\big)\, ds \,, \qquad \varphi_0(t) = y_0 \,. \tag{2.8}$$

Note that φ_n is continuous for all $n \in \mathbb{N}$. We show that the sequence of functions $\{\varphi_k\}_{k=1}^{\infty}$ converges to a solution to (2.7).

Claim 1: For all $n \in \mathbb{N} \cup \{0\}$,

$$\big|\varphi_n(t) - y_0\big| \leqslant k \qquad \forall\, t \in I_h \,. \tag{2.9}$$

Proof of claim 1: We prove claim 1 by induction. Clearly (2.9) holds for $n = 0$. Now suppose that (2.9) holds for $n = N$. Then for $n = N + 1$ and $t \in I_h$,

$$\big|\varphi_{N+1}(t) - y_0\big| \leqslant \Big| \int_{t_0}^{t} f\big(s, \varphi_N(s)\big)\, ds \Big| \leqslant M|t - t_0| \leqslant k \,.$$

Claim 2: For all $n \in \mathbb{N} \cup \{0\}$,

$$\max_{t \in I_h} \big|\varphi_{n+1}(t) - \varphi_n(t)\big| \leqslant k^{n+1} \,.$$

Proof of claim 2: Let $e_{n+1}(t) = \varphi_{n+1}(t) - \varphi_n(t)$. Using (2.8) and the mean value theorem, we find that

$$e_{n+1}(t) = \int_{t_0}^{t} \big[f\big(s, \varphi_{n+1}(s)\big) - f\big(s, \varphi_n(s)\big) \big]\, ds = \int_{t_0}^{t} f_y\big(s, \xi_n(s)\big) e_n(s)\, ds$$

for some function ξ_n satisfying $\big|\xi_n(t) - y_0\big| \leqslant k$ in I_h (by claim 1); thus with ϵ_n denoting $\max\limits_{t \in I_h} \big|e_n(t)\big|$,

$$\epsilon_{n+1} \leqslant k \epsilon_n \qquad \forall\, n \in \mathbb{N} \,;$$

thus

$$\epsilon_{n+1} \leqslant k\epsilon_{n-1} \leqslant k^2\epsilon_{n-1} \leqslant \cdots \leqslant k^n\epsilon_1 = k^n \max_{t\in I_h}\Big|\int_{t_0}^t f(s,y_0)\,ds\Big| \leqslant Mhk^n = k^{n+1}\,.$$

Claim 3: The sequence of functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ converges for each $t \in I_h$.

Proof of claim 3: Note that

$$\varphi_{n+1}(t) = y_0 + \sum_{j=0}^{n}\big[\varphi_{j+1}(t) - \varphi_j(t)\big]\,.$$

For each fixed $t \in I_h$, the series $\sum\limits_{j=0}^{\infty}\big[\varphi_{j+1}(t) - \varphi_j(t)\big]$ converges absolutely (by claim 2 with the comparison test). Therefore, $\{\varphi_n(t)\}_{n=1}^{\infty}$ converges for each $t \in I_h$.

Claim 4: The limit function φ is continuous in I_h.

Proof of Claim 4: Let $\varepsilon > 0$ be given. Choose $\delta = \dfrac{\varepsilon}{2M}$. Then if $t_1, t_2 \in I_h$ satisfying $|t_1 - t_2| < \delta$, we must have

$$\big|\varphi_{n+1}(t_1) - \varphi_{n+1}(t_2)\big| \leqslant \Big|\int_{t_1}^{t_2} f\big(s,\varphi_n(s)\big)\,ds\Big| \leqslant M|t_1 - t_2| < \frac{\varepsilon}{2}\,.$$

Passing to the limit as $n \to \infty$, we conclude that

$$\big|\varphi(t_1) - \varphi(t_2)\big| \leqslant \frac{\varepsilon}{2} < \varepsilon \quad \forall\, t_1, t_2 \in I_h \text{ and } |t_1 - t_2| < \delta$$

which implies that φ is continuous in I_h.

Claim 5: The limit function φ satisfies $\varphi(t) = y_0 + \displaystyle\int_{t_0}^t f\big(s,\varphi(s)\big)\,ds$ for all $t \in I_h$.

Proof of claim 5: It suffices to show that

$$\lim_{n\to\infty}\int_{t_0}^t f\big(s,\varphi_n(s)\big)\,ds = \int_{t_0}^t f\big(s,\varphi(s)\big)\,ds \qquad \forall\, t \in I_h\,.$$

Let $\varepsilon > 0$ be given. Choose $N > 0$ such that $\dfrac{k^{N+2}}{1-k} < \varepsilon$. Then by claim 2 and the mean value theorem, for $n \geqslant N$,

$$\begin{aligned}\Big|\int_{t_0}^t f\big(s,\varphi_n(s)\big)\,ds - \int_{t_0}^t f\big(s,\varphi(s)\big)\,ds\Big| &= \Big|\int_{t_0}^t f_y\big(s,\xi(s)\big)\big[\varphi_n(s) - \varphi(s)\big]\,ds\Big| \\ &\leqslant M\Big|\int_{t_0}^t \sum_{j=n}^{\infty}\big|\varphi_{j+1}(s) - \varphi_j(s)\big|\,ds\Big| \leqslant M|t - t_0|\sum_{j=N}^{\infty} k^{j+1} \leqslant \frac{k^{N+2}}{1-k} < \varepsilon\,.\end{aligned}$$

Claim 6: $y = \varphi(t)$ is a solution to (2.7).

Proof of claim 6: Since φ is continuous, by the fundamental theorem of Calculus,

$$\frac{d}{dt}\Big[y_0 + \int_{t_0}^t f\big(s,\varphi(s)\big)\,ds\Big] = f\big(t,\varphi(t)\big)$$

which implies that $\varphi'(t) = f\big(t,\varphi(t)\big)$. Moreover, $\varphi(0) = y_0$; thus $y = \varphi(t)$ is a solution to (2.7).

Uniqueness: Suppose that $y = \psi(t)$ is a solution to the ODE (2.7) in the time interval I_h such that $\big|\psi(t) - y_0\big| \leqslant k$ in I_h. Let $\vartheta = \varphi - \psi$. Then ϑ solves

$$\vartheta' = f(t,\varphi) - f(t,\psi) = f_y\big(t,\xi(t)\big)\vartheta \qquad \vartheta(t_0) = 0$$

for some ξ in between φ and ψ satisfying $|\xi(t) - y_0| \leqslant k$. Integrating in t over the time interval $[t_0, t]$ we find that

$$\vartheta(t) = \int_{t_0}^{t} f_y(s,\xi(s))\vartheta(s)\,ds\,.$$

(a) If $t > t_0$,

$$|\vartheta(t)| \leqslant \Big|\int_{t_0}^{t} \big|f_y(s,\xi(s))\big||\vartheta(s)|\,ds\Big| \leqslant M\int_{t_0}^{t} |\vartheta(s)|\,ds\,;$$

thus the fundamental theorem of Calculus implies that

$$\frac{d}{dt}\Big(e^{-Mt}\int_{t_0}^{t} \big|\vartheta(s)\big|\,ds\Big) = e^{-Mt}\Big(|\vartheta(t)| - M\int_{t_0}^{t} |\vartheta(s)|\Big) \leqslant 0\,.$$

Therefore,

$$e^{-Mt}\int_{t_0}^{t} \big|\vartheta(s)\big|\,ds \leqslant e^{-Mt_0}\int_{t_0}^{t_0} \big|\vartheta(s)\big|\,ds = 0$$

which implies that $\vartheta(t) = 0$ for all $t \in I_h$.

(b) If $t < t_0$,

$$|\vartheta(t)| \leqslant \Big|\int_{t_0}^{t} \big|f_y(s,\xi(s))\big||\vartheta(s)|\,ds\Big| \leqslant M\int_{t}^{t_0} |\vartheta(s)|\,ds = -M\int_{t_0}^{t} |\vartheta(s)|\,ds\,;$$

thus the fundamental theorem of Calculus implies that

$$\frac{d}{dt}\Big(e^{Mt}\int_{t_0}^{t} \big|\vartheta(s)\big|\,ds\Big) = e^{Mt}\Big(|\vartheta(t)| + M\int_{t_0}^{t} |\vartheta(s)|\Big) \leqslant 0\,.$$

Therefore,

$$e^{-Mt}\int_{t_0}^{t} \big|\vartheta(s)\big|\,ds \geqslant e^{-Mt_0}\int_{t_0}^{t_0} \big|\vartheta(s)\big|\,ds = 0$$

which implies that $\vartheta(t) = 0$ for all $t \in I_h$.

Finally, we need to argue if it is possible to have a solution $y = y(t)$ in the time interval I_h but $\big|y(t) - y_0\big| > k$ for some $t \in I_h$. If so, by the continuity of the solution there must be some $t_1 \in I_h$ such that $\big|y(t_1) - y_0\big| = k$. We then can solve the ODE

$$\psi' = f(t,\psi) \qquad \psi(t_1) = y(t_1)\,,$$

and the previous argument implies that there is a time interval $\widetilde{I}$ in which the solution is unique. Since $y = \varphi(t)$ is a solution in the time interval I_h, we must have $\varphi = \psi$ in $I_h \cap \widetilde{I}$. This concludes the uniqueness of the solution to (2.7). □

Remark 2.27. In the proof of the existence and the uniqueness theorem, the condition that f_y is continuous is not essential. This condition can be replaced by that f is (local) Lipschitz in its second variable; that is, there exists $L > 0$ such that

$$\big|f(t, y_1) - f(t, y_2)\big| \leqslant L|y_1 - y_2|\,.$$

Example 2.28. Solve the initial value problem $y' = 2t(1 + y)$ with initial data $y(0) = 0$ using the Picard iteration.

Recall the Picard iteration

$$\varphi_{k+1}(t) = \int_0^t 2s\big(1 + \varphi_k(s)\big)\,ds \qquad \text{with } \varphi_0(t) = 0. \tag{2.10}$$

Then $\varphi_1(t) = \int_0^t 2s\,ds = t^2$, and $\varphi_2(t) = \int_0^t 2s(1 + s^2)\,ds = t^2 + \frac{t^4}{2}$, and then $\varphi_3(t) = \int_0^t 2s\big(1 + s^2 + \frac{s^4}{2}\big)\,ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$. To see a general rule, we observe that $\varphi_k(t)$ must be a polynomial of the form

$$\varphi_k(t) = \sum_{j=1}^{k} a_j t^{2j}\,,$$

and $\varphi_{k+1}(t) = \varphi_k(t) + a_{k+1}t^{2(k+1)}$. Therefore, we only need to determine the coefficients a_k in order to find the solution. Note that using (2.10) we have

$$\sum_{j=1}^{k+1} a_j t^{2j} = \int_0^t 2s\big(1 + \sum_{j=1}^{k} a_j t^{2j}\big)\,ds = t^2 + \sum_{j=1}^{k} \frac{2a_j}{2j+2} t^{2j+2} = t^2 + \sum_{j=2}^{k+1} \frac{a_{j-1}}{j} t^{2j}\,;$$

thus the comparison of coefficients implies that $a_1 = 1$, $a_j = \frac{a_{j-1}}{j}$. Therefore,

$$a_k = \frac{a_{k-1}}{k} = \frac{a_{k-2}}{k(k-1)} = \cdots = \frac{a_1}{k(k-1)\cdots 2} = \frac{1}{k!}$$

which implies that $\varphi_k(t) = \sum\limits_{j=1}^{k} \frac{t^{2j}}{j!} = \sum\limits_{j=0}^{k} \frac{t^{2j}}{j!} - 1$. Using the Maclaurin series of the exponential function, we find that $\varphi_k(t)$ converges to $e^{t^2} - 1$. The function $\varphi(t) = e^{t^2} - 1$ is indeed a solution of the ODE under consideration.

Remark 2.29. Usually the Picard iteration can be used to find the solution to those ODEs that we can solve using the techniques introduced in Section 2.1, 2.2 and 2.6.

2.9 First Order Difference Equations

Definition 2.30 (Difference Equations)**.** A k-th order difference equation is of the form

$$y_{n+k} = f(k, n, y_{n+k-1}, y_{n+k-2}, \cdots, y_n) \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,. \tag{2.11}$$

The initial condition for a k-th order difference equation is some given numbers $y_0, y_1, \cdots, y_{k-1}$. A solution to the difference equation with given initial data is a sequence $\{y_k\}_{k=0}^{\infty}$ that satisfies the difference equation.

The difference equation (2.11) is said to be linear if f is linear in $(y_{n+k-1}, y_{n+k-2}, \cdots, y_n)$. It is called nonlinear if it is no linear. The difference equation (2.11) is said to be autonomous if f is independent of n and k.

A constant solution to an autonomous difference equation is called an equilibrium solution.

2.9.1 Linear first order difference equation

• Consider $y_{n+1} = \rho_n y_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then $y_n = y_0 \prod_{k=0}^{n-1} \rho_k$.

Equilibrium solution: Solve $c = \rho_n c$ for all $n \in \mathbb{N}$.

1. If ρ_n depends on n, then the only equilibrium solution is 0.
2. If ρ_n is independent of n; that is, $\rho_n = \rho$ for all $n \in \mathbb{N}$, then
   (a) if $\rho \neq 1$, 0 is the only equilibrium solution.
   (b) if $\rho = 1$, any constant is a equilibrium solution.

   Moreover,
$$\lim_{n\to\infty} y_n = \begin{cases} 0 & \text{if } |\rho| < 1\,, \\ y_0 & \text{if } \rho = 1\,, \\ \text{DNE} & \text{otherwise}\,; \end{cases}$$
   thus $y = 0$ is an asymptotically stable solution if $|\rho| < 1$.

• Next, consider a more complicated first order linear difference equation: $y_{n+1} = \rho_n y_n + b_n$.

$$\begin{aligned} y_n &= \rho_{n-1} y_{n-1} + b_{n-1} = \rho_{n-1}\big(\rho_{n-2} y_{n-2} + b_{n-2}\big) + b_{n-1} = \rho_{n-1}\rho_{n-2} y_{n-2} + \rho_{n-1} b_{n-2} + b_{n-1} \\ &= \cdots = y_0 \prod_{k=0}^{n-1} \rho_k + \Big(b_{n-1} + \rho_{n-1} b_{n-2} + \cdots + \rho_{n-1}\cdots\rho_1 b_0\Big)\,. \end{aligned}$$

If $\rho_n = \rho$ and $b_n = b$ for all $n \in \mathbb{N} \cup \{0\}$, then

$$y_n = \rho^n y_0 + \big(b + \rho b + \cdots + \rho^{n-1} b\big) = \begin{cases} \rho^n\Big(y_0 + \dfrac{b}{\rho - 1}\Big) + \dfrac{b}{1-\rho} & \text{if } \rho \neq 1\,, \\ \rho^n y_0 + nb & \text{if } \rho = 1\,. \end{cases} \tag{2.12}$$

In general, there is no equilibrium solution. However, if $\rho_n = \rho$ and $b_n = b$ for all $n \in \mathbb{N} \cup \{0\}$, then $y = \dfrac{b}{1-\rho}$ is an equilibrium solution if $\rho \neq 1$. Using (2.12), we find that $\dfrac{b}{1-\rho}$ is an asymptotically stable solution if $|\rho| < 1$.

2.9.2 Nonlinear first order difference equations

• Consider $y_{n+1} = \rho y_n\Big(1 - \dfrac{y_n}{k}\Big)$. Noting that using Euler's method to discretize the logistic equation $\dfrac{dy}{dt} = ry\big(1 - \frac{y}{K}\big)$, we have

$$\frac{u_{n+1} - u_n}{\Delta t} = r u_n\Big(1 - \frac{u_n}{K}\Big) \Rightarrow u_{n+1} = (1 + r\Delta t) u_n\Big(1 - \frac{r\Delta t}{K(1 + r\Delta t)} u_n\Big)\,.$$

Letting $x_n = y_n/k$, we have

$$x_{n+1} = \rho x_n(1 - x_n)\,. \tag{2.13}$$

Equilibrium solution: Solving $c = \rho c(1-c)$, we obtain that $c = 0$ and $c = 1 - \dfrac{1}{\rho}$ are equilibrium solutions to (2.13).

Definition 2.31. A equilibrium solution $y = c$ is called an asymptotically stable equilibrium solution to the difference equation $y_{n+1} = f(y_n)$ if there exists $\delta > 0$ such that if $y_0 \in (c-\delta, c+\delta)$, the solution y_n approaches c as $n \to \infty$.

To check the (linear) stability of these equilibrium solution, we rely on the following

Theorem 2.32. *Let f be a twice differentiable function, and c be a solution to $c = f(c)$. Then c is an asymptotically stable equilibrium to $y_{n+1} = f(y_n)$ if $\left|f'(c)\right| < 1$.*

Proof. By that f is twice continuously differentiable,

$$\lim_{\delta\to 0^+}\Big(|f'(c)| + \frac{\delta}{2}\max_{x\in[c-\delta,c+\delta]}|f''(x)|\Big) = |f'(c)| < 1\,;$$

thus there exists $\delta > 0$ such that $\rho(\delta) \equiv |f'(c)| + \dfrac{\delta}{2}\max\limits_{x\in[c-\delta,c+\delta]}|f''(x)| < 1$. Fix such $\delta > 0$ and let $\rho \equiv \rho(\delta)$. If $0 < |y_n - c| < \delta$, then Taylor's theorem implies that for some d_n in between y_n and c,

$$y_{n+1} = f(y_n) = f(c) + f'(c)(y_n - c) + \frac{1}{2}f''(d_n)(y_n-c)^2 = c + f'(c)(y_n - c) + \frac{1}{2}f''(d_n)(y_n - c)^2$$

which further implies that

$$|y_{n+1} - c| \leqslant |f'(c)||y_n - c| + \frac{1}{2}\max_{x\in(c-\delta,c+\delta)}|f''(x)||y_n - c|^2 \leqslant \rho\delta < \delta\,.$$

In other words, if $|y_0 - c| < \delta$, then $|y_n - c| < \delta$ for all $n \in \mathbb{N}$. As a consequence,

$$|y_{n+1} - c| \leqslant |f'(c)||y_n - c| + \frac{1}{2}\max_{x\in(c-\delta,c+\delta)}|f''(x)||y_n - c|^2 \leqslant \rho|y_n - c|\,;$$

hence $|y_n - c| \leqslant \rho^n|y_0 - c|$ which implies that $y_n \to c$ as $n \to \infty$ if $|y_0 - c| < \delta$. □

Remark 2.33. Theorem 2.32 only provides a sufficient condition for determining the (linear) stability for the difference equation $y_{n+1} = f(y_n)$ near the equilibrium solution. When the derivative of f at the equilibrium solution is 1, no conclusion can be drawn and it has to be discussed case by case.

Let $f(x) = \rho x(1-x) = \rho x - \rho x^2$. Then $f'(x) = \rho - 2\rho x$.

The equilibrium solution $y_n = 0$: Since $f'(0) = \rho$, the equilibrium solution $c = 0$ is asymptotically stable if $|\rho| < 1$.

The equilibrium solution $y_n = 1 - \dfrac{1}{\rho}$: Since $f'\big(1-\rho^{-1}\big) = 2-\rho$, the equilibrium solution $c = 1-\rho^{-1}$ is asymptotically stable if $|2-\rho| < 1$ or equivalently, $1 < \rho < 3$.

Exchange of stability: As ρ increases (from 0), the equilibrium solution $y = 0$ becomes unstable when $\rho = 1$.

Other cases:

1. If $\rho = 3.2$, there is a "periodic" solution of period 2.

2. If $\rho = 3.5$, there is a "periodic" solution of period 4.

3 Second Order Linear Equations

Definition 3.1. A second order ordinary differential equation has the form

$$f\Big(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\Big) = 0 \tag{3.1}$$

for some given function f. The ODE (3.1) is called ***linear*** if the function f takes the form

$$f\Big(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\Big) = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y - G(t)\,,$$

where P is a function which never vanishes for all $t > 0$. The ODE (3.1) is called ***nonlinear*** if it is not linear. The functions P, Q, R are called the ***coefficients*** of the ODE, and G is called the ***forcing*** of the ODE. The ***initial condition*** for (3.1) is $\big(y(t_0), y'(t_0)\big) = (y_0, y_1)$.

3.1 Homogeneous Equations with Constant Coefficients

Definition 3.2. The ODE (3.1) is called ***homogeneous*** if $g \equiv 0$, otherwise it is called ***non-homogeneous***. When $g \not\equiv 0$, the term $g(t)$ in (3.1) is called the non-homogeneous term.

In this section, we consider homogeneous second order linear ODE with constant coefficients

$$Py'' + Qy' + Ry = 0\,,$$

where P, Q, R are independent of t. Since $P \neq 0$, the ODE reduces to

$$y'' + by' + cy = 0\,. \tag{3.2}$$

Let λ be the solution to the equation $\lambda^2 + b\lambda + c = 0$.

1. Suppose that there are two distinct real roots λ_1 and λ_2. Then

$$\Big(\frac{d}{dt} - \lambda_1\Big)\Big(\frac{d}{dt} - \lambda_2\Big)y = 0\,.$$

Therefore, if $z = (\frac{d}{dt} - \lambda_2)y$, then $(\frac{d}{dt} - \lambda_1)z = 0$ which further implies that $z = c_1e^{\lambda_1 t}$ for some constant c_1. Then

$$\begin{aligned} y' - \lambda_2 y = c_1e^{\lambda_1 t} \Rightarrow (e^{-\lambda_2 t}y)' = c_1e^{(\lambda_1-\lambda_2)t} \Rightarrow e^{-\lambda_2 t}y = \frac{c_1}{\lambda_1 - \lambda_2}e^{(\lambda_1-\lambda_2)t} + c_2 \\ \Rightarrow y = \frac{c_1}{\lambda_1 - \lambda_2}e^{\lambda_1 t} + c_2e^{\lambda_2 t}\,. \end{aligned}$$

In other words, a solution to the ODE (3.2) is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ if λ_1 and λ_2 are distinct real roots of $\lambda^2 + b\lambda + c = 0$.

2. Suppose that there is a real double root λ. Then the argument show that y satisfies

$$y' - \lambda y = c_1e^{\lambda t} \Rightarrow (e^{-\lambda t}y)' = c_1 \Rightarrow e^{-\lambda t}y = c_1t + c_2 \Rightarrow y = c_1te^{\lambda t} + c_2e^{\lambda t}\,.$$

In other words, a solution to the ODE (3.2) is a linear combination of $te^{\lambda t}$ and $e^{\lambda t}$ if λ is the real double root of $\lambda^2 + b\lambda + c = 0$.

Question: What happened if there are complex roots for $\lambda^2 + b\lambda + c = 0$?

Definition 3.3. The characteristic equation for the ODE (3.2) is $\lambda^2 + b\lambda + c = 0$.

Another way to derive the characteristic equations: Consider $y'' + by' + cy = 0$. Let $y' = z$. Then

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}.$$

Write $\boldsymbol{x} = [y, z]^{\mathrm{T}}$ and $A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$. Then $\boldsymbol{x}' = A\boldsymbol{x}$.

Suppose that $A = P\Lambda P^{-1}$ for some diagonal matrix Λ; that is, A is diagonalizable (with eigenvectors of A form the columns of P and eigenvalues forms the diagonal entry of Λ), then $P^{-1}\boldsymbol{x}' = \Lambda P^{-1}\boldsymbol{x}$. Letting $\boldsymbol{u} = P^{-1}\boldsymbol{x}$, then $\boldsymbol{u}' = \Lambda\boldsymbol{u}$ or equivalently,

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Therefore, $u_1' = \lambda_1 u_1$ and $u_2' = \lambda_2 u_2$ that further imply that $u_1 = c_1 e^{\lambda_1 t}$ and $u_2 = c_2 e^{\lambda_2 t}$. Since $\boldsymbol{x} = P\boldsymbol{u}$, we conclude that y is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.

What are eigenvalues of A? Let λ be an eigenvalue of A. Then

$$\begin{vmatrix} -\lambda & 1 \\ -c & -b-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + b\lambda + c = 0$$

which is the characteristic equation. Therefore, eigenvalues of A are the roots of the characteristic equation for the ODE (3.2).

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

In this section, we consider the ODE

$$L[y] = y'' + py' + qy = 0$$

with initial condition $y(t_0) = y_0$ and $y'(t_0) = y_1$.

Theorem 3.4. *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, y'(t_0) = y_1,$$

where p, q and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \varphi(t)$ of this problem, and the solution exists throughout the interval I.

In the following, we assume that p, q are continuous in the interval of interests.

Theorem 3.5 (Principle of Superposition)**.** *If $y = \varphi_1$ and $y = \varphi_2$ are two solutions of the differential equation*

$$L[y] = y'' + py' + qy = 0, \tag{3.3}$$

then the linear combination $c_1\varphi_1 + c_2\varphi_2$ is also a solution for any values of the constants c_1 and c_2. In other words, the collection of solutions to (3.3) is a vector spaces.

Question: Given two solutions $y = \varphi_1$ and $y = \varphi_2$ of the differential equation (3.3), can the solution to the differential equation

$$L[y] = y'' + py' + qy = 0 \quad \text{with initial condition } y(t_0) = y_0 \text{ and } y'(t_0) = y_1 \tag{3.4}$$

can be written as a linear combination of φ_1 and φ_2 (for whatever given initial data)? If this is true, then

$$\text{the } \underbrace{\text{vector spaces consisting of solutions}}_{\text{called the solution space}} \text{ to (3.3) is two-dimensional,}$$

and $\{\varphi_1, \varphi_2\}$ is a basis of the solution space of (3.3).

How do one know if the solution to (3.4) can be written as a linear combination of φ_1 and φ_2? Suppose that for given initial data y_0, y_1 there exist constants c_1, c_2 such that $y(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$ is a solution to (3.4). Then

$$\begin{bmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

So for any given initial data (y_0, y_1) the solution to (3.4) can be written as a linear combination of φ_1 and φ_2 if the matrix $\begin{bmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{bmatrix}$ is non-singular. This induces the following

Definition 3.6. Let φ_1 and φ_2 be two differentiable functions. The ***Wronskian*** or ***Wronskian determinant*** of φ_1 and φ_2 at point t_0 is the number

$$W(\varphi_1, \varphi_2)(t_0) = \det\Big(\begin{bmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{bmatrix} \Big) = \varphi_1(t_0)\varphi_2'(t_0) - \varphi_2(t_0)\varphi_1'(t_0).$$

The collection of functions $\{\varphi_1, \varphi_2\}$ is called a ***fundamental set*** of equation (3.3) if $W(\varphi_1, \varphi_2)(t) \neq 0$ for some t in the interval of interest.

Moreover, we also establish the following

Theorem 3.7. *Suppose that $y = \varphi_1$ and $y = \varphi_2$ are two solutions of the ODE* (3.3). *Then for any arbitrarily given (y_0, y_1), the solution to the ODE*

$$L[y] = y'' + py' + qy = 0 \quad \textit{with initial condition } y(t_0) = y_0 \textit{ and } y'(t_0) = y_1,$$

can be written as a linear combination of φ_1 and φ_2 if and only if the Wronskian of φ_1 and φ_2 at t_0 does not vanish.

Theorem 3.8. *Let φ_1 and φ_2 be solutions to the differential equation* (3.3) *satisfying the initial conditions $\big(\varphi_1(t_0), \varphi_1'(t_0)\big) = (1, 0)$ and $\big(\varphi_2(t_0), \varphi_2'(t_0)\big) = (0, 1)$. Then $\{\varphi_1, \varphi_2\}$ is a fundamental set of equation* (3.3)*, and for any (y_0, y_1), the solution to* (3.4) *can be written as $y = y_0\varphi_1 + y_1\varphi_2$.*

Theorem 3.9. *If a complex-valued function $u + iv$ is a solution to* (3.3)*, so is its real part u and imaginary part v.*

Next, suppose that φ_1, φ_2 are solutions to (3.3) and $W(\varphi_1, \varphi_2)(t_0) \neq 0$. We would like to know if $\{\varphi_1, \varphi_2\}$ can be used to construct solutions to the differential equation

$$L[y] = y'' + py' + qy = 0 \quad \text{with initial condition } y(t_1) = y_0 \text{ and } y'(t_1) = y_1 \tag{3.5}$$

for some $t_1 \neq t_0$. In other words, we would like to know if $W(\varphi_1, \varphi_2)(t_1)$ vanishes or not. This question is answered by the following

Theorem 3.10 (Abel)**.** *Let φ_1 and φ_2 be two solutions of* (3.3) *in which p, q are continuous in an open interval I, and the Wronskian $W(\varphi_1, \varphi_2)$ does not vanish at $t_0 \in I$. Then*

$$W(\varphi_1, \varphi_2)(t) = W(\varphi_1, \varphi_2)(t_0) \exp\Big(- \int_{t_0}^t p(s)ds \Big) .$$

In particular, $W(\varphi_1, \varphi_2)(t)$ is never zero for all $t \in I$.

Proof. Since φ_1 and φ_2 are solutions to (3.3), we have

$$\varphi_1''(t) + p(t)\varphi_1'(t) + q(t)\varphi_1(t) = 0 , \tag{3.6a}$$

$$\varphi_2''(t) + p(t)\varphi_2'(t) + q(t)\varphi_2(t) = 0 . \tag{3.6b}$$

Computing (3.6*b*) $\times \varphi_1$ − (3.6*a*) $\times \varphi_2$, we obtain that

$$(\varphi_2\varphi_1'' - \varphi_1\varphi_2'') + p(\varphi_2\varphi_1' - \varphi_1\varphi_2') = 0$$

Therefore, letting $W = \varphi_2\varphi_1' - \varphi_1\varphi_2'$ be the Wronskian of φ_1 and φ_2. Then $W' + pW = 0$; thus

$$W(t) = W(t_0) \exp\Big(- \int_{t_0}^t p(s)ds \Big) .$$

Since p is continuous on $[t_0, t]$ (or $[t, t_0]$), the integral $\displaystyle\int_{t_0}^t p(s)ds$ is finite; thus $W(t) \neq 0$. □

3.3 Complex Roots of the Characteristic Equation

Consider again the 2nd order linear homogeneous ordinary differential equation

$$y'' + by' + cy = 0 \tag{3.2}$$

where b and c are both constants. Suppose that the characteristic equation $r^2 + br + c = 0$ has two complex roots $\lambda \pm i\mu$. We expect that the solution to (3.2) can be written as a linear combination of $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$.

What is $e^{i\mu t}$? The Euler identity says that $e^{i\theta} = \cos\theta + i \sin\theta$; thus

$$e^{(\lambda\pm i\mu)t} = e^{\lambda t}\big[\cos(\mu t) \pm i \sin(\mu t)\big] .$$

By Theorem 3.9, we see that $\varphi_1(t) = e^{\lambda t}\cos(\mu t)$ and $e^{\lambda t}\sin(\mu t)$ are also solutions to (3.2).

Checking the linear independence: Computing the Wronskian of φ_1 and φ_2, we find that

$$W(\varphi_1, \varphi_2)(t) = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ e^{\lambda t}\big(\lambda\cos(\mu t) - \mu\sin(\mu t)\big) & e^{\lambda t}\big(\lambda\sin(\mu t) + \mu\cos(\mu t)\big) \end{vmatrix} = \mu e^{\lambda t}$$

which is non-zero if $\mu \neq 0$. Therefore, any solution to (3.2) can be written as a linear combination of φ_1 and φ_2 if $b^2 - 4c < 0$.

Example 3.11. Consider the motion of an object attached to a spring. The dynamics is described by the 2nd order ODE:

$$m\ddot{x} = -kx - r\dot{x}\,, \tag{3.7}$$

where m is the mass of the object, k is the Hooke constant of the spring, and r is the friction coefficient.

1. If $r^2 - 4mk > 0$: There are two distinct negative roots $\dfrac{-r \pm \sqrt{r^2 - 4mk}}{2m}$ to the characteristic equation, and the solution of (3.7) can be written as

$$x(t) = C_1 \exp\Big(\frac{-r + \sqrt{r^2 - 4mk}}{2m}t\Big) + C_2 \exp\Big(\frac{-r - \sqrt{r^2 - 4mk}}{2m}t\Big)\,.$$

   The solution $x(t)$ approaches zero as $t \to \infty$.

2. If $r^2 - 4mk = 0$: There is one negative double root $\dfrac{-r}{2m}$ to the characteristic equation, and the solution of (3.7) can be written as

$$x(t) = C_1 \exp\Big(\frac{-rt}{2m}\Big) + C_2 t \exp\Big(\frac{-rt}{2m}\Big)\,.$$

   The solution $x(t)$ approaches zero as $t \to \infty$.

3. If $r^2 - 4mk < 0$: There are two complex roots $\dfrac{-r \pm i\sqrt{4mk - r^2}}{2m}$ to the characteristic equation, and the solution of (3.7) can be written as

$$x(t) = C_1 e^{-\frac{rt}{2m}} \cos\big(\frac{\sqrt{4mk - r^2}}{2m}t\big) + C_2 e^{-\frac{rt}{2m}} \sin\big(\frac{\sqrt{4mk - r^2}}{2m}t\big)\,.$$

   (a) If $r = 0$, the motion of the object is periodic with period $\dfrac{4m\pi}{\sqrt{4mk - r^2}}$, and is called **simple harmonic motion**.

   (b) If $r > 0$, the object oscillates about the equilibrium point ($x = 0$) but approaches to zero exponentially.

3.4 Repeated Roots; Reduction of Order

In Section 3.1 we have discussed the case that the characteristic equation of the homogeneous equation with constant coefficients

$$y'' + by' + cy = 0 \tag{3.2}$$

has one double root. We recall that in such case $b^2 = 4c$, and $\varphi_1(t) = \exp\big(\frac{-bt}{2}\big)$, $\varphi_2(t) = t\exp\big(\frac{-bt}{2}\big)$ together form a fundamental set of (3.2).

Suppose that we are given a solution $\varphi_1(t)$. Since (3.2) is a second order equation, there should be two linearly independent solutions. One way of finding another solution, using information that φ_1 provides, is **the variation of constant**: suppose another solution is given by $\varphi_2(t) = v(t)\varphi_1(t)$. Then

$$v''\varphi_1 + 2v'\varphi_1' + v\varphi_1'' + b\big(v'\varphi_1 + v\varphi_1'\big) + cv\varphi_1 = 0\,.$$

Since $y = \varphi_1(t)$ verifies (3.2), we find that

$$v''\varphi_1 + 2v'\varphi_1' + bv'\varphi_1 = 0\,;$$

thus using $\varphi_1(t) = \exp\big(\frac{-bt}{2}\big)$ we obtain $v''\varphi_1 = 0$. Since φ_1 never vanishes, $v''(t) = 0$ for all t. Therefore, $v(t) = C_1 t + C_2$ for some constant C_1 and C_2. Therefore, another solution to (3.2), when $b^2 = 4c$, is $\varphi_2(t) = t\exp\big(\frac{-bt}{2}\big)$.

The idea of the variation of constant can be generalize to homogeneous equations with variable coefficients. Suppose that we have found a solution $y = \varphi_1(t)$ to the second order homogeneous equation

$$y'' + p(t)y' + q(t)y = 0\,. \tag{3.8}$$

Assume that another solution is given by $y = v(t)\varphi_1(t)$. Then v satisfies

$$v''\varphi_1 + 2v'\varphi_1' + v\varphi_1'' + p(v'\varphi_1 + v\varphi_1') + qv\varphi_1 = 0\,.$$

By the fact that φ_1 solves (3.8), we find that v satisfies

$$v''\varphi_1 + 2v'\varphi_1' + pv'\varphi_1 = 0 \quad \text{or equivalently,} \quad v''\varphi_1 + v'(2\varphi_1' + p\varphi_1) = 0\,. \tag{3.9}$$

The equation above can be solved (for v') using the method of integrating factor, and is essentially a first order equation.

Let P be an anti-derivative of p. If $\varphi_1(t) \neq 0$ for all $t \in I$, then (3.9) implies that

$$\big(\varphi_1^2(t)e^{P(t)}v'(t)\big)' = 0 \quad \Rightarrow \quad \varphi_1^2(t)e^{P(t)}v'(t) = C \quad \Rightarrow \quad \varphi_1^2(t)v'(t) = Ce^{-P(t)} \qquad \forall\, t \in I\,.$$

As a consequence,

$$W(\varphi_1,\varphi_2)(t) = \begin{vmatrix} \varphi_1(t) & v(t)\varphi_1(t) \\ \varphi_1'(t) & v'(t)\varphi_1(t) + v(t)\varphi_1'(t) \end{vmatrix} = \begin{vmatrix} \varphi_1(t) & 0 \\ \varphi_1'(t) & v'(t)\varphi_1(t) \end{vmatrix} = \varphi_1^2(t)v'(t) = Ce^{-P(t)} \neq 0$$

which implies that $\{\varphi_1, v\varphi_1\}$ is indeed a fundamental set of (3.8).

Example 3.12. Given that $y = \varphi_1(t) = \dfrac{1}{t}$ is a solution of

$$2t^2y'' + 3ty' - y = 0 \qquad \text{for } t > 0\,, \tag{3.10}$$

find a fundamental set of the equation.

Suppose another solution is given by $y = v(t)\varphi_1(t) = v(t)/t$. Then (3.9) implies that v satisfies

$$v''(t)\frac{1}{t} + v'\Big(-\frac{2}{t^2} + \frac{3}{2t}\frac{1}{t}\Big) = 0\,.$$

Therefore, $v'' = \dfrac{v'}{2t}$; thus $v'(t) = C_1\sqrt{t}$ which further implies that $v(t) = \dfrac{2}{3}C_1 t^{\frac{3}{2}} + C_2$. Therefore, one solution to (3.10) is

$$y = \frac{2}{3}C_1\sqrt{t} + C_2\frac{1}{t}$$

which also implies that $y = \varphi_2(t) = \sqrt{t}$ is a solution to (3.10). Note that the Wronskian

$$W(\varphi_1,\varphi_2)(t) = \begin{vmatrix} \dfrac{1}{t} & \sqrt{t} \\ -\dfrac{1}{t^2} & \dfrac{1}{2\sqrt{t}} \end{vmatrix} = \frac{3}{2}t^{-\frac{3}{2}} \neq 0 \quad \text{for } t > 0\,; \tag{3.11}$$

thus $\{\varphi_1, \varphi_2\}$ is indeed a fundamental set of (3.10).

3.5 Nonhomogeneous Equations

In this section, we focus on solving the second order nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t)\,. \tag{3.12}$$

Definition 3.13. A ***particular solution*** to (3.12) is a twice differentiable function validating (3.12). In other words, a particular solution is a solution of (3.12). The space of ***complementary solutions*** to (3.12) is the collection of solutions to the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0\,. \tag{3.13}$$

Let $y = Y(t)$ be a particular solution to (3.12). If $y = \varphi(t)$ is another solution to (3.12), then $y = \varphi(t) - Y(t)$ is function in the space of complementary solutions to (3.12). By Theorem 3.8, there exist two function φ_1 and φ_2 such that $y = \varphi_j(t)$, $j = 1, 2$, are linearly independent solutions to (3.13), and $\varphi(t) - Y(t) = C_1\varphi_1(t) + C_2\varphi_2(t)$ for some constants C_1 and C_2. This observation shows the following

Theorem 3.14. *The general solution of the nonhomogeneous equation* (3.12) *can be written in the form*

$$y = \varphi(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + Y(t)\,,$$

where $\{\varphi_1, \varphi_2\}$ *is a fundamental set of* (3.13)*,* C_1 *and* C_2 *are arbitrary constants, and* $y = Y(t)$ *is a particular solution of the nonhomogeneous equation* (3.12)*.*

General strategy of solving nonhomogeneous equation (3.12):

1. Find the space of complementary solution to (3.12); that is, find the general solution $y = C_1\varphi_1(t) + C_2\varphi_2(t)$ of the homogeneous equation (3.13).

2. Find a particular solution $y = Y(t)$ of the nonhomogeneous equation (3.12).

3. Apply Theorem 3.14.

3.5.1 Method of Variation of Parameters

This method can be used to solve a nonhomogeneous ODE when one solution to the corresponding homogeneous equation is known.

Consider

$$y'' + p(t)y' + q(t)y = g(t)\,. \tag{3.12}$$

Suppose that we are given one solution $y = \varphi_1(t)$ to the corresponding homogeneous euqation

$$y'' + p(t)y' + q(t)y = 0\,. \tag{3.13}$$

Using the procedure in Section 3.4, we can find another solution $y = \varphi_2(t)$ to (3.13) so that $\{\varphi_1, \varphi_2\}$ forms a fundamental set of (3.13). Our goal next is to obtain a particular solution to (3.12).

Suppose a particular solution $y = Y(t)$ can be written as the product of two functions u and φ_1; that is, $Y(t) = u(t)\varphi_1(t)$. Then similar computations as in Section 3.4 show that

$$u''\varphi_1 + u'(2\varphi_1' + p\varphi_1) = g \quad \Rightarrow \quad (\varphi_1^2 e^P u')' = \varphi_1 e^P g\,,$$

where P is an anti-derivative of p. Therefore,

$$\varphi_1^2(t)e^{P(t)}u'(t) = \int \varphi_1(t)e^{P(t)}g(t)\,dt\,,$$

and further computations yield that

$$u(t) = \int \frac{\displaystyle\int \varphi_1(t)e^{P(t)}g(t)\,dt}{\varphi_1^2(t)e^{P(t)}}\,dt\,.$$

Therefore, a particular solution is of the form

$$Y(t) = \varphi_1(t)\int \frac{\displaystyle\int \varphi_1(t)e^{P(t)}g(t)\,dt}{\varphi_1^2(t)e^{P(t)}}\,dt\,. \tag{3.14}$$

Example 3.15. As in Example 3.12, let $y = \varphi_1(t) = \dfrac{1}{t}$ be a given solution to

$$2t^2y'' + 3ty' - y = 0 \qquad \text{for } t > 0\,, \tag{3.10}$$

Suppose that we are looking for solutions to

$$2t^2y'' + 3ty' - y = 2t^2 \qquad \text{for } t > 0\,. \tag{3.15}$$

Using (3.14) (noting that in this case $g(t) = 1$), we know that a particular solution is given by

$$Y(t) = \frac{1}{t}\int \frac{\displaystyle\int t^{-1}e^{3/2\log t}dt}{t^{-2}e^{3/2\log t}}\,dt = \frac{1}{t}\int \Big(t^{\frac{1}{2}}\int t^{\frac{1}{2}}dt\Big)dt = \frac{2}{9}t^2\,.$$

Therefore, combining with the fact that $\varphi_2(t) = \sqrt{t}$ is a solution to (3.10), we find that a general solution to (3.15) is given by

$$y = \frac{C_1}{t} + C_2\sqrt{t} + \frac{2}{9}t^2\,.$$

Let $\{\varphi_1, \varphi_2\}$ be a fundamental set of (3.13) (here φ_2 is either given or obtained using the procedure in previous section), we can also look for a particular solution to (3.12) of the form

$$Y(t) = c_1(t)\varphi_1(t) + c_2(t)\varphi_2(t)\,.$$

Plugging such Y in (3.12)), we find that

$$c_1''\varphi_1 + c_1'(2\varphi_1' + p\varphi_1) + c_2''\varphi_2 + c_2'(2\varphi_2' + p\varphi_2) = g\,. \tag{3.16}$$

Since we increase the degree of freedom (by adding another function c_2), we can impose an additional constraint. Assume that the additional constraint is

$$c_1'\varphi_1 + c_2'\varphi_2 = 0\,. \tag{3.17}$$

Then $c_1''\varphi_1 + c_2''\varphi_2 = -c_1'\varphi_1' - c_2'\varphi_2'$; thus (3.16) reduces to

$$c_1'\varphi_1' + c_2'\varphi_2' = g\,. \tag{3.18}$$

Solving (3.17) and (3.18), we find that

$$c_1' = \frac{-g\varphi_2}{W(\varphi_1,\varphi_2)} \qquad \text{and} \qquad c_2' = \frac{g\varphi_1}{W(\varphi_1,\varphi_2)}\,.$$

The discussion above establishes the following

Theorem 3.16. *If the function p, q and g are continuous in an open interval I, and $\{\varphi_1,\varphi_2\}$ be a fundamental set of the ODE* (3.13)*. Then a particular solution to* (3.12) *is*

$$Y(t) = -\varphi_1(t)\int_{t_0}^{t}\frac{g(s)\varphi_2(s)}{W(\varphi_1,\varphi_2)(s)}\,ds + \varphi_2(t)\int_{t_0}^{t}\frac{g(s)\varphi_1(s)}{W(\varphi_1,\varphi_2)(s)}\,ds\,, \tag{3.19}$$

where $t_0 \in I$ can be arbitrarily chosen, and the general solution to (3.12) *is*

$$y = C_1\varphi_1(t) + C_2\varphi_2(t) + Y(t)\,.$$

Example 3.17. Given two solutions $\varphi_1(t) = \dfrac{1}{t}$ and $\varphi_2(t) = \sqrt{t}$ to the ODE

$$2t^2y'' + 3ty' - y = 0 \qquad \text{for } t > 0\,. \tag{3.10}$$

To solve

$$2t^2y'' + 3ty' - y = 2t^2 \qquad \text{for } t > 0\,, \tag{3.15}$$

we use (3.19) and (3.11) to obtain that a particular solution to (3.15) is given by

$$Y(t) = -\frac{1}{t}\int\frac{\sqrt{t}}{\frac{3}{2}t^{-3/2}}\,dt + \sqrt{t}\int\frac{t^{-1}}{\frac{3}{2}t^{-3/2}}\,dt = \frac{2}{9}t^2\,.$$

Therefore, a general solution to (3.15) is given by

$$y = \frac{C_1}{t} + C_2\sqrt{t} + \frac{2}{9}t^2\,.$$

3.5.2 Method of Undetermined Coefficients

In addition to the method of variation of parameters, some tricks can be made to solve nonhomogeneous equations with constant coefficients and special forcing functions. In this sub-section, we focus on solving

$$y'' + by' + cy = g(t)\,. \tag{3.20}$$

Suppose that λ_1 and λ_2 are two roots of $r^2 + br + c = 0$ (λ_1 and λ_2 could be identical or complex-valued). Then (3.20) can be written as

$$\Big(\frac{d}{dt} - \lambda_1\Big)\Big(\frac{d}{dt} - \lambda_2\Big)y(t) = g(t)\,.$$

Letting $y' - \lambda_2 y = z$, we have $z' - \lambda_1 z = g(t)$; thus

$$z(t) = e^{\lambda_1 t} \int e^{-\lambda_1 t} g(t)\, dt\,.$$

Solving for y we obtain that

$$y(t) = e^{\lambda_2 t} \int \Big(e^{(\lambda_1 - \lambda_2)t} \int e^{-\lambda_1 t} g(t)\, dt \Big)\, dt\,. \tag{3.21}$$

Consider the following three types of forcing function g:

1. $g(t) = p_n(t) e^{\alpha t}$ for some polynomial $p_n(t) = a_n t^n + \cdots + a_1 t + a_0$ of degree n: note that

$$\int e^{\gamma t} t^k\, dt = \begin{cases} \dfrac{1}{\gamma} e^{\gamma t} t^k - \dfrac{k}{\gamma} \displaystyle\int e^{\gamma t} t^{k-1} dt & \text{if } \gamma \neq 0\,, \\ \dfrac{1}{k+1} t^{k+1} + C & \text{if } \gamma = 0\,. \end{cases} \tag{3.22}$$

   Therefore, in this case a particular solution is of the form

$$Y(t) = t^s (A_n t^n + \cdots + A_1 t + A_0) e^{\alpha t}$$

   for some unknown s and coefficients $A_i's$, and we need to determine the values of these unknowns.

   (a) If $\lambda_1 \neq \alpha$ and $\lambda_2 \neq \alpha$, then $s = 0$.

   (b) If $\lambda_1 = \alpha$ but $\lambda_2 \neq \alpha$, then $s = 1$.

   (c) If $\lambda_1 = \lambda_2 = \alpha$, then $s = 2$.

2. $g(t) = p_n(t) e^{\alpha t} \cos(\beta t)$ or $g(t) = p_n(t) e^{\alpha t} \sin(\beta t)$ for some polynomial p_n of degree n and $\beta \neq 0$: note that (3.22) also holds for $\gamma \in \mathbb{C}$. Therefore, in this case we assume that a particular solution is of the form

$$Y(t) = t^s \Big[(A_n t^n + \cdots + A_1 t + A_0) e^{\alpha t} \cos(\beta t) + (B_n t^n + \cdots + B_1 t + B_0) e^{\alpha t} \sin(\beta t) \Big]$$

   for some unknown s and coefficients $A_i's$, $B_i's$, and we need to determine the values of these unknowns.

   (a) If $\lambda_1, \lambda_2 \in \mathbb{R}$, then $s = 0$.

   (b) If $\lambda_1, \lambda_2 \notin \mathbb{R}$; that is, $\lambda_1 = \gamma + i\delta$ and $\lambda_2 = \gamma - i\delta$ for some $\delta \neq 0$:

      (1) If $\lambda_1 = \gamma + i\delta$ and $\lambda_2 = \gamma - i\delta$ for some $\gamma \neq \alpha$ or $\delta \neq \pm\beta$, then $s = 0$.

      (2) If $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then $s = 1$.

Example 3.18. Find a particular solution of $y'' - 3y' - 4y = 3e^{2t}$.

Since the roots of the characteristic equation $r^2 - 3r - 4$ are different from -1, we expect that a particular solution to the ODE above is of the form Ae^{2t}. Solving for A, we find that $A = -\dfrac{1}{2}$; thus a particular solution is $Y(t) = -\dfrac{1}{2} e^{2t}$.

Example 3.19. Find a particular solution of $y'' - 3y' - 4y = 2\sin t$.

Since the roots of $r^2 - 3r - 4 = 0$ are real, we expect that a particular solution is of the form $Y(t) = A\cos t + B\sin t$ for some constants A, B to be determined. In other words, we look for A, B such that

$$(A\cos t + B\sin t)'' - 3(A\cos t + B\sin t)' - 4(A\cos t + B\sin t) = 2\sin t\,.$$

By expanding the derivatives and comparing the coefficients, we find that (A, B) satisfies

$$\begin{cases} 3A - 5B = 2\,, \\ 5A + 3B = 0\,, \end{cases}$$

and the solution to the equation above is $(A, B) = \big(\frac{3}{17}, \frac{-5}{17}\big)$. Therefore, a particular solution is

$$Y(t) = \frac{3}{17}\cos t - \frac{5}{17}\sin t\,.$$

Example 3.20. Find a particular solution of $y'' - 3y' - 4y = -8e^t\cos 2t$.

Since the roots of $r^2 - 3r - 4 = 0$ are real, we expect that a particular solution is of the form $Y(t) = Ae^t\cos 2t + Be^t\sin 2t$ for some constants A, B to be determined. In other words, we look for A, B such that

$$(Ae^t\cos 2t + Be^t\sin 2t)'' - 3(Ae^t\cos 2t + Be^t\sin 2t)' - 4(Ae^t\cos 2t + Be^t\sin 2t) = -8e^t\cos 2t\,.$$

By expanding the derivatives,

	$(e^t\cos 2t)''$	$(e^t\sin 2t)''$	$(e^t\cos 2t)'$	$(e^t\sin 2t)'$	$e^t\cos 2t$	$e^t\sin t$
$e^t\cos 2t$	-3	4	1	2	1	0
$e^t\sin 2t$	-4	-3	-2	1	0	1

thus

$$\begin{aligned} -3A + 4B - 3A - 6B - 4A &= -8\,, \\ -4A - 3B + 6A - 3B - 4B &= 0\,. \end{aligned}$$

Therefore, $(A, B) = (\frac{10}{13}, \frac{2}{13})$; thus a particular solution is

$$Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t\,.$$

Example 3.21. Find a particular solution of $y'' - 3y' - 4y = 2e^{-t}$.

Since one of the roots of the characteristic equation $r^2 - 3r - 4$ is -1, we expect that a particular solution to the ODE above is of the form Ate^{-t} for some constant A to be determined. In other words, we look for A such that

$$(Ate^{-t})'' - 3(Ate^{-t})' - 4Ate^{-t} = 2e^{-t}\,.$$

By expanding the derivatives, we find that $-5A = 2$ which implies that $A = -\frac{2}{5}$. Therefore, a particular solution is given by $Y(t) = -\frac{2}{5}te^{-t}$.

How about if we forget what s is? - By trial and error! Starting from $s = 0$. If a particular of the form with $s = 0$ cannot be found, then try $s = 1$, and so on.

Example 3.22. Find a particular solution of $y'' - 4y' + 5y = -2e^{2t}\sin t$.

We first look for a particular solution of the form $Y(t) = Ae^{2t}\cos t + Be^{2t}\sin t$, and find that this leads to that $0 = e^{2t}\sin t$ which is impossible. Therefore, we look for a particular solution of the form $Y(t) = t(Ae^{2t}\cos t + Be^{2t}\sin t)$. Note that

	$(te^{2t}\cos t)''$	$(te^{2t}\sin t)''$	$(te^{2t}\cos t)'$	$(te^{2t}\sin t)'$	$te^{2t}\cos t$	$te^{2t}\sin t$
$te^{2t}\cos t$	3	4	2	1	1	0
$te^{2t}\sin t$	-4	3	-1	2	0	1
$e^{2t}\cos t$	4	2	1	0	0	0
$e^{2t}\sin t$	-2	4	0	1	0	0

thus by assuming this form of particular solutions we find that

$$\begin{aligned} 3A + 4B - 8A - 4B + 5A &= 0\,, \\ -4A + 3B + 4A - 8B + 5B &= 0\,, \\ 4A + 2B - 4A &= 0\,, \\ -2A + 4B - 4B &= -2\,. \end{aligned}$$

Therefore, $(A, B) = (1, 0)$, and a particular solution is $Y(t) = te^t\cos t$.

We also note that using (3.19) we find another particular solution

$$y = \big(t - \frac{\sin 2t}{2}\big)e^t\cos t + \frac{\cos 2t}{2}e^t\sin t = te^t\cos t - \frac{1}{2}e^t\sin t\,.$$

If the forcing g is the sum of functions of different types, the construction of a particular solution relies on the following

Theorem 3.23. *If $y = \varphi_j(t)$ is a particular solution to the ODE*

$$y'' + p(t)y' + q(t)y = g_j(t)$$

for all $j = 1, \cdots n$, then the function $y = \sum\limits_{j=1}^{n}\varphi_j(t)$ is a particular to the ODE

$$y'' + p(t)y' + q(t)y = g(t) \equiv \sum_{j=1}^{n} g_j(t)\,.$$

Example 3.24. Find a particular solution of $y'' - 3y' - 4y = 3e^{2t} - 8e^t\cos 2t + 2e^{-t}$.

By Example 3.18, 3.20 and 3.21, a particular solution to the ODE above is

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t - \frac{2}{5}te^{-t}\,.$$

3.6 Mechanical and Electrical Vibrations

We have been discussing the motion of an object attached to a spring without external force in Example 3.11. Now we explain what if there are presence of external forcings. We consider

$$m\ddot{x} = -kx - r\dot{x} + g(t)\,, \tag{3.23}$$

where m, k, r are positive constants. We remark that the term $-r\dot{x}$ is sometimes called a ***damping*** or ***resistive*** force, and r is called the ***damping coefficient***.

1. **Undamped Free Vibrations**: This case refers to that $g \equiv 0$ and $r = 0$. The solution to (3.23) is then
$$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega t = R\cos(\omega_0 t - \phi)\,,$$
where $R = \sqrt{C_1^2 + C_2^2}$ is called the ***amplitude***, $\omega_0 = \sqrt{\dfrac{k}{m}}$ is called the ***natural frequency*** and $\phi = \arctan\dfrac{C_2}{C_1}$ is called the ***phase angle***. The period of this vibration is $T = \dfrac{2\pi}{\omega_0}$.

2. **Dampled Free Vibrations**: This case refers to that $g \equiv 0$ and $r > 0$. The solution to (3.23) is then
$$x(t) = C_1 e^{-\frac{rt}{2m}} \cos\mu t + C_2 e^{-\frac{rt}{2m}} \sin\mu t = Re^{-\frac{rt}{2m}} \cos(\mu t - \phi)\,,$$
where $R = \sqrt{C_1^2 + C_2^2}$, $\mu = \dfrac{\sqrt{4km - r^2}}{2m}$, and $\phi = \arctan\dfrac{C_2}{C_1}$. Here μ is called the ***quasi frequency***, and we note that
$$\frac{\mu}{\omega_0} = \Big(1 - \frac{r^2}{4km}\Big)^{\frac{1}{2}} \approx 1 - \frac{r^2}{8km}\,,$$
where the last approximation holds if $\dfrac{r^2}{4km} \ll 1$. The period of this vibration $\dfrac{2\pi}{\mu}$ is called the ***quasi period***.

   (a) **Critical damped**: In this case, $r^2 = 4km$.

   (b) **Overdamped**: This case refers to that $r^2 > 4km$, and in this case the attached object pass the equilibrium at most once and does not oscillate about equilibrium.

3. **Forced Vibrations with Damping**: We only consider
$$m\ddot{x} + r\dot{x} + kx = F_0 \cos\omega t \tag{3.24}$$
for some non-zero r, F_0 and ω. Let $\{\varphi_1, \varphi_2\}$ be a fundamental set of the corresponding homogeneous equation of (3.24). From the discussion above, φ_1 and φ_2 both decay to zero (die out) as $t \to \infty$. Using what we learn from the method of undetermined coefficients, the general solution to (3.24) is
$$x = \underbrace{C_1\varphi_1(t) + C_2\varphi_2(t)}_{\equiv x_c(t)} + \underbrace{A\cos\omega t + B\sin\omega t}_{\equiv X(t)}\,,$$

where C_1 and C_2 are chosen to satisfy the initial condition, and A and B are some constants so that $X(t) = A\cos\omega t + B\sin\omega t$ is a particular solution to (3.24). The part $x_c(t)$ is called the ***transient solution*** and it decays to zero (die out) as $t \to \infty$; thus as $t \to \infty$, one sees that only a steady oscillation with the same frequency as the external force remains in the motion. $x = X(t)$ is called the ***steady state solution*** or the ***forced response***.

Since $x = X(t)$ is a particular solution to (3.24), (A, B) satisfies

$$(k - \omega^2 m)A + r\omega B = F_0\,,$$
$$-r\omega A + (k - \omega^2 m)B = 0\,;$$

thus with ω_0 denoting the natural frequency; that is, $\omega_0 = \dfrac{k}{m}$, we have

$$(A, B) = \Big(\frac{F_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + r^2\omega^2}, \frac{F_0 r\omega}{m^2(\omega_0^2 - \omega^2)^2 + r^2\omega^2}\Big)\,.$$

Let $\alpha = \dfrac{\omega}{\omega_0}$, and $\Gamma = \dfrac{r^2}{mk}$. Then

$$(A, B) = \frac{F_0}{k}\Big(\frac{1 - \alpha^2}{(1 - \alpha^2)^2 + \Gamma\alpha^2}, \frac{\sqrt{\Gamma}\alpha}{(1 - \alpha^2)^2 + \Gamma\alpha^2}\Big)\,;$$

thus

$$X(t) = R\cos(\omega t - \phi)\,,$$

where with Δ denoting the number $\sqrt{(1 - \alpha^2)^2 + \Gamma\alpha^2}$, we have

$$R = \sqrt{A^2 + B^2} = \frac{F_0}{k\Delta} \quad \text{and} \quad \phi = \arccos\frac{1 - \alpha^2}{\Delta}\,.$$

Then if $\alpha \ll 1$, $R \approx \dfrac{F_0}{k}$ and $\phi \approx 0$, while if $\alpha \gg 1$, $R \ll 1$ and $\phi \approx \pi$.

In the intermediate region, some α, called $\alpha_{\max}$, maximize the amplitude R. Then $\alpha_{\max}$ minimize $(1 - \alpha^2)^2 + \Gamma\alpha^2$ which implies that $\alpha_{\max}$ satisfies

$$\alpha_{\max}^2 = 1 - \frac{\Gamma}{2}$$

and, when $\Gamma < 1$, the corresponding maximum amplitude $R_{\max}$ is

$$R_{\max} = \frac{F_0}{k}\frac{1}{\sqrt{\Gamma}\sqrt{1 - \Gamma/4}} \approx \frac{F_0}{k\sqrt{\Gamma}}\Big(1 + \frac{\Gamma}{8}\Big)\,,$$

where the last approximation holds if $\Gamma \ll 1$. If $\Gamma > 2$, the maximum of R occurs at $\alpha = 0$ (and the maximum amplitude is $R_{\max} = \dfrac{F_0}{k}$).

For lightly damped system; that is, $r \ll 1$ (which implies that $\Gamma \ll 1$), the maximum amplitude $R_{\max}$ is closed to a very large number $\dfrac{F_0}{k\sqrt{\Gamma}}$. In this case $\alpha_{\max} \approx 1$, and this implies that the frequency $\omega_{\max}$, where the maximum of R occurs, is very close to ω_0. We call such a phenomena (that $R_{\max} \gg 1$ when $\omega \approx \omega_0$) ***resonance***. In such a case, $\alpha_{\max} \approx 1$; thus $\phi = \dfrac{\pi}{2}$ which means the response occur $\dfrac{\pi}{2}$ later than the peaks of the excitation.

4. **Forced Vibrations without Damping**:

   (a) When $r = 0$, if $\omega \neq \omega_0$, then general solution to (3.24) is

$$x = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t\,,$$

   where C_1 and C_2 depends on the initial data. We are interested in the case that $x(0) = x'(0) = 0$. In this case,

$$C_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad C_2 = 0\,,$$

   so the solution to (3.24) (with initial condition $x(0) = x'(0) = 0$) is

$$x = \frac{F_0}{m(\omega_0^2 - \omega^2)}\big(\cos \omega t - \cos \omega_0 t\big) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{\omega_0 - \omega}{2} t \sin \frac{\omega_0 + \omega}{2} t\,.$$

   When $\omega \approx \omega_0$, $R = \dfrac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \dfrac{\omega_0 - \omega}{2} t$ presents a slowly varying sinusoidal amplitude. This type of motion, possessing a periodic variation of amplitude, is called a ***beat***.

   (b) When $r = 0$ and $\omega = \omega_0$, the general solution to (3.24) is

$$x = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t\,.$$

4 Higher Order Linear Equations

4.1 General Theory of n-th Order Linear Equations

An n-th order linear ordinary differential equations is an equation of the form

$$P_n(t)\frac{d^n y}{dt^n} + P_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_1\frac{dy}{dt} + P_0(t)y = G(t)\,,$$

where P_n is never zero in the time interval of interest. Divide both sides by $P_n(t)$, we obtain

$$L[y] = \frac{d^n y}{dt} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)\,. \tag{4.1}$$

Suppose that $p_j \equiv 0$ for all $0 \leqslant j \leqslant n-1$. Then to determine y, it requires n times integration and each integration produce an arbitrary constant. Therefore, we expect that to determine the solution y to (4.1) uniquely, it requires n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \cdots, \quad y^{(n-1)}(t_0) = y_{n-1}\,, \tag{4.2}$$

where t_0 is some point in an open interval I, and $y_0, y_1, \cdots, y_{n-1}$ are some given constants.

Equation (4.1) is called ***homogeneous*** if $g \equiv 0$.

Theorem 4.1. *If the functions $p_0, \cdots, p_{n-1}$ and g are continuous on an open interval I, then there exists exactly one solution $y = \varphi(t)$ of the differential equation* (4.1) *with initial condition* (4.2)*, where t_0 is any point in I. This solution exists throughout the interval I.*

Definition 4.2. Let $\{\varphi_1, \cdots, \varphi_n\}$ be a collection of n differentiable functions defined on an open interval I. The Wronskian of $\varphi_1, \varphi_2, \cdots, \varphi_n$ at $t_0 \in I$, denoted by $W(\varphi_1, \cdots, \varphi_n)(t_0)$, is the number

$$W(\varphi_1, \cdots, \varphi_n)(t_0) = \begin{vmatrix} \varphi_1(t_0) & \varphi_2(t_0) & \cdots & \varphi_n(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) & \cdots & \varphi_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(t_0) & \varphi_2^{(n-1)}(t_0) & \cdots & \varphi_n^{(n-1)}(t_0) \end{vmatrix}.$$

Theorem 4.3. *Let $y = \varphi_1(t)$, $y = \varphi_2(t)$, $\cdots$, $y = \varphi_n(t)$ be solutions to the homogeneous equation*

$$L[y] = \frac{d^n y}{dt} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0\,. \tag{4.3}$$

Then the Wronskian of $\varphi_1, \varphi_2, \cdots, \varphi_n$ satisfies

$$\frac{d}{dt}W(\varphi_1, \cdots, \varphi_n)(t) + p_{n-1}(t)W(\varphi_1, \cdots, \varphi_n)(t) = 0\,.$$

Proof. By the differentiation of the determinant, we find that

$$\frac{d}{dt}W(\varphi_1, \cdots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2'(t_0) & \cdots & \varphi_n' \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \cdots & \varphi_n^{(n-2)} \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix}$$

$$= \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2'(t_0) & \cdots & \varphi_n' \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \cdots & \varphi_n^{(n-2)} \\ -p_{n-1}\varphi_1^{(n-1)} - \cdots - p_0\varphi_1 & -p_{n-1}\varphi_2^{(n-1)} - \cdots - p_0\varphi_2 & \cdots & -p_{n-1}\varphi_n^{(n-1)} - \cdots - p_0\varphi_n \end{vmatrix}$$

$$= -p_{n-1}W(\varphi_1, \cdots, \varphi_n)\,.$$

□

Theorem 4.4. *Suppose that the functions $p_0, \cdots, p_{n-1}$ are continuous on an open interval I. If $y = \varphi_1(t)$, $y = \varphi_2(t)$, $\cdots$, $y = \varphi_n(t)$ are solutions to the homogeneous equation* (4.3) *and the Wronskian $W(\varphi_1, \cdots, \varphi_n)(t) \neq 0$ for at least one point $t \in I$, then every solution of* (4.3) *can be expressed as a linear combination of $\varphi_1, \cdots, \varphi_n$.*

Proof. Let $y = \varphi(t)$ be a solution to (4.3), and suppose that $W(\varphi_1, \cdots, \varphi_n)(t_0) \neq 0$. Define $(y_0, y_1, \cdots, y_{n-1}) = \big(\varphi(t_0), \varphi'(t_0), \cdots, \varphi^{(n-1)}(t_0)\big)$, and let $C_1, \cdots, C_n \in \mathbb{R}$ be the solution to

$$\begin{bmatrix} \varphi_1(t_0) & \varphi_2(t_0) & \cdots & \varphi_n(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) & \cdots & \varphi_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(t_0) & \varphi_2^{(n-1)}(t_0) & \cdots & \varphi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

We note that the system above has a unique solution since $W(\varphi_1, \cdots, \varphi_n)(t_0) \neq 0$.

Claim: $\varphi(t) = C_1\varphi_1(t) + \cdots + C_n\varphi_n(t)$.

Proof of Claim: Note that $y = \varphi(t)$ and $y = C_1\varphi_1(t) + \cdots + C_n\varphi_n(t)$ are both solutions to (4.3) satisfying the same initial condition. Therefore, by Theorem 4.1 the solution is unique, so the claim is concluded. □

Definition 4.5. A collection of solutions $\{\varphi_1, \cdots, \varphi_n\}$ to (4.3) is called a fundamental set of equation (4.3) if $W(\varphi_1, \cdots, \varphi_n)(t) \neq 0$ for some t in the interval of interest.

4.1.1 Linear Independence of Functions

Recall that in a vector space $(\mathcal{V}, +, \cdot)$ over scalar field $\mathbb{F}$, a collection of vectors $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ is called linearly dependent if there exist constants $c_1, \cdots, c_n$ in $\mathbb{F}$ such that $\prod_{i=1}^{n} c_i \equiv c_1 \cdot c_2 \cdot \cdots \cdot c_{n-1} \cdot c_n \neq 0$ and

$$c_1 \cdot \boldsymbol{v}_1 + \cdots + c_n \cdot \boldsymbol{v}_n = \mathbf{0}\,.$$

If no such $c_1, \cdots, c_n$ exists, $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ is called linearly independent. In other words, $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\} \subseteq \mathcal{V}$ is linearly independent if and only if

$$c_1 \cdot \boldsymbol{v}_1 + \cdots + c_n \cdot \boldsymbol{v}_n = \mathbf{0} \quad \Leftrightarrow \quad c_1 = c_2 = \cdots = c_n = 0\,.$$

Now let $\mathcal{V}$ denote the collection of all $(n-1)$-times differentiable functions defined on an open interval I. Then $(\mathcal{V}, +, \cdot)$ clearly is a vector space over $\mathbb{R}$. Given $\{f_1, \cdots, f_n\} \subseteq \mathcal{V}$, we would like to determine the linear dependence or independence of the n-functions $\{f_1, \cdots, f_n\}$. Suppose that

$$c_1 f_1(t) + \cdots + c_n f_n(t) = 0 \qquad \forall\, t \in I\,.$$

Since each f_j are $(n-1)$-times differentiable, we have for $1 \leqslant k \leqslant n-1$,

$$c_1 f_1^{(k)}(t) + \cdots + c_n f_n^{(k)}(t) = 0 \qquad \forall\, t \in I\,.$$

In other words, $c_1, \cdots, c_n$ satisfy

$$\begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & & \vdots & \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \forall\, t \in I\,.$$

If there exists $t_0 \in I$ such that the matrix $\begin{bmatrix} f_1(t_0) & f_2(t_0) & \cdots & f_n(t_0) \\ f_1'(t_0) & f_2'(t_0) & \cdots & f_n'(t_0) \\ \vdots & & \vdots & \\ f_1^{(n-1)}(t_0) & f_2^{(n-1)}(t_0) & \cdots & f_n^{(n-1)}(t_0) \end{bmatrix}$ is non-singular, then $c_1 = c_2 = \cdots = c_n = 0$. Therefore, a collection of solutions $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of equation (4.3) if and only if $\{\varphi_1, \cdots, \varphi_n\}$ is linearly independent.

4.1.2 The Homogeneous Equations - Reduction of Orders

Suppose that $y = \varphi_1(t)$ is a solution to (4.3). Now we look for a function v such that $y = v(t)\varphi_1(t)$ is also a solution to (4.3). The derivative of this v satisfies an $(n-1)$-th order homogeneous ordinary differential equation.

Example 4.6. Suppose that we are given $y = \varphi_1(t) = e^t$ as a solution to

$$(2-t)y''' + (2t-3)y'' - ty' + y = 0 \quad \text{for} \quad t < 2\,. \tag{4.4}$$

Suppose that $y = v(t)e^t$ is also a solution to (4.4). Then

$$(2-t)(v'''e^t + 3v''e^t + 3v'e^t + ve^t) + (2t-3)(v''e^t + 2v'e^t + ve^t) - t(v'e^t + ve^t) + ve^t = 0$$

which implies that v satisfies

$$(2-t)v''' + \big[3(2-t) + (2t-3)\big]v'' + \big[3(2-t) + 2(2t-3) - t\big]v' = 0$$

or equivalently, with u denoting v'',

$$(2-t)u' + (3-t)u = 0\,.$$

Solving the ODE above, we find that $u(t) = C_1(2-t)e^{-t}$ for some constant C_1; thus

$$v(t) = C_3 + C_2 t + C_1 e^{-t} - C_1(t+1)e^{-t} = C_3 + C_2 t - C_1 t e^{-t}\,.$$

Therefore, a fundamental set of (4.4) is $\{e^t, te^t, t\}$.

4.1.3 The Nonhomogeneous Equations

Let $y = Y_1(t)$ and $y = Y_2(t)$ be solutions to (4.1). Then $y = Y_1(t) - Y_2(t)$ is a solution to the homogeneous equation (4.3); thus if $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of (4.3), then

$$Y_1(t) - Y_2(t) = C_1\varphi_1(t) + \cdots + C_n\varphi_n(t)\,.$$

Therefore, we establish the following theorem which is similar to Theorem 3.14.

Theorem 4.7. *The general solution of the nonhomogeneous equation* (4.1) *can be written in the form*

$$y = \varphi(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + \cdots + C_n\varphi_n(t) + Y(t)\,,$$

where $\{\varphi_1, \cdots, \varphi_n\}$ *is a fundamental set of* (4.3), $C_1, \cdots, C_n$ *are arbitrary constants, and* $y = Y(t)$ *is a particular solution of the nonhomogeneous equation* (4.1).

In general, in order to solve (4.1), we follow the procedure listed below:

1. Find the space of complementary solution to (4.3); that is, find the general solution $y = C_1\varphi_1(t) + C_2\varphi_2(t) + \cdots + C_n\varphi_n$ of the homogeneous equation (4.3).

2. Find a particular solution $y = Y(t)$ of the nonhomogeneous equation (4.1).

3. Apply Theorem 4.7.

4.2 Homogeneous Equations with Constant Coefficients

We now consider the n-th order linear homogeneous ODE with constant coefficients

$$L[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0\,, \tag{4.5}$$

where a_j's are constants for all $j \in \{0, 1, \cdots, n-1\}$. Suppose that $r_1, r_2, \cdots, r_n$ are solutions to the characteristic equation of (4.5)

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0\,.$$

Then (4.5) can be written as

$$\Big(\frac{d}{dt} - r_1\Big)\Big(\frac{d}{dt} - r_2\Big)\cdots\Big(\frac{d}{dt} - r_n\Big)y = 0$$

1. If the characteristic equation of (4.5) has distinct roots, then

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \cdots + C_n e^{r_n t}\,. \tag{4.6}$$

Reason: Let $z_1 = \big(\frac{d}{dt} - r_2\big)\cdots\big(\frac{d}{dt} - r_n\big)y$. Then $z_1' - r_1 z_1 = 0$; thus $z_1(t) = c_1 e^{r_1 t}$.
Let $z_2 = \big(\frac{d}{dt} - r_3\big)\cdots\big(\frac{d}{dt} - r_n\big)y$. Then $z_2' - r_2 z_2 = c_1 z_1$; thus using the method of integrating factors, we find that

$$\frac{d}{dt}\big(e^{-r_2 t}z_2\big) = c_1 e^{(r_1 - r_2)t} \quad\Rightarrow\quad z_2(t) = \frac{c_1}{r_1 - r_2}e^{r_1 t} + c_2 e^{r_2 t}\,. \tag{4.7}$$

Repeating the process, we conclude (4.6).

How about if there are complex roots? Suppose that $r_1 = a + bi$ and $r_2 = a - bi$, then the Euler identity implies that, by choosing complex c_1 and c_2 in (4.7), we find that

$$z_2(t) = c_1 e^{at}\cos bt + c_2 e^{at}\sin bt$$

for some constants c_1 and c_2. Therefore, suppose that we have complex roots $a_k \pm b_k i$ for $k = 1, \cdots, \ell$ and real roots $r_{2\ell+1}, \cdots, r_n$. Then the general solution to (4.6) is

$$\begin{aligned} y(t) = {} & C_1 e^{a_1 t}\cos b_1 t + C_2 e^{a_1 t}\sin b_1 t + \cdots + C_{2\ell-1}e^{a_\ell t}\cos b_\ell t + C_{2\ell}e^{a_\ell t}\sin b_\ell t \\ & + C_{2\ell+1}e^{r_{2\ell+1}t} + \cdots + C_n e^{r_n t}\,. \end{aligned}$$

2. If the characteristic equation of (4.5) has repeated roots, we group the roots in such a way that $r_1 = r_2 = \cdots = r_\ell$ and so on; that is, repeated roots appear in a successive order. Then the implication in (4.7) is modified to

$$\frac{d}{dt}\big(e^{-r_2 t}z_2\big) = c_1 e^{(r_1 - r_2)t} = c_1 \quad\Rightarrow\quad z_2(t) = (c_1 t + c_2)e^{r_1 t}\,.$$

(a) Suppose that $r_3 = r_2 = r_1 = r$. Letting $z_3 = \left(\frac{d}{dt} - r_4\right) \cdots \left(\frac{d}{dt} - r_n\right)y$, we find that

$$z_3' - rz_3 = (c_1 t + c_2)e^{rt};$$

thus the method of integrating factor implies that

$$\frac{d}{dt}\left(e^{-rt}z_3\right) = c_1 t + c_2 \quad \Rightarrow \quad z_3(t) = \left(\frac{c_1}{2}t^2 + c_2 t + c_3\right)e^{rt}.$$

(b) Suppose that $r_1 = r_2 = r$ and $r_3 \neq r_2$. Letting $z_3 = \left(\frac{d}{dt} - r_4\right) \cdots \left(\frac{d}{dt} - r_n\right)y$, we find that

$$z_3' - r_3 z_3 = (c_1 t + c_2)e^{rt};$$

thus the method of integrating factor implies that

$$\frac{d}{dt}\left(e^{-r_3 t}z_3\right) = (c_1 t + c_2)e^{(r-r_3)t} \quad \Rightarrow \quad z_3(t) = \left(\widetilde{c_1}t + \widetilde{c_2}\right)e^{rt} + c_3 e^{r_3 t}.$$

From the discussion above, we "conjecture" that if r_j's are roots of the characteristic equation of (4.5) with multiplicity n_j (so that $n_1 + \cdots + n_k = n$), then the general solution to (4.5) is

$$y(t) = \sum_{j=1}^{k} p_j(t)e^{r_j t},$$

where $p_j(t)$'s are some polynomials of degree $n_j - 1$. Note that in each p_j there are n_j constants to be determined by the initial conditions.

If there are repeated complex roots, say $r_1 = a + bi$ and $r_2 = a - bi$ with $n_1 = n_2$. Then p_1 and p_2 are polynomials of degree n_1; thus by adjusting constants in the polynomials properly, we find that

$$p_1(t)e^{r_1 t} + p_2(t)e^{r_2 t} = \widetilde{p}_1(t)e^{at}\cos bt + \widetilde{p}_2(t)e^{at}\sin bt.$$

In other words, if r_j are real roots of the characteristic equation of (4.5) with multiplicity n_j and $a_k \pm ib_k$ are complex roots of the characteristic equation of (4.5) with multiplicity m_k (so that $\sum_j n_j + \sum_k 2m_k = n$), then the general solution to (4.5) is

$$y(t) = \sum_j p_j(t)e^{r_j t} + \sum_k e^{a_k t}\left(q_k^1(t)\cos b_k t + q_k^2(t)\sin b_k t\right),$$

where $p_j(t)$'s are some polynomials of degree $n_j - 1$ and q_k^1, q_k^2's are some polynomials of degree $m_k - 1$.

Example 4.8. Find the general solution of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0.$$

The roots of the characteristic equation is $r = \pm 1$, $r = 2$ and $r = -3$; thus the general solution to the ODE above is

$$y = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-3t}.$$

If we are looking for a solution to the ODE above satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$ and $y'''(0) = -1$, then C_1, C_2, C_3, C_4 have to satisfy

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}.$$

Solving the linear system above, we find that the solution solving the ODE with the given initial data is

$$y = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}.$$

Example 4.9. Find the general solution of

$$y^{(4)} - y = 0\,.$$

Also find the solution that satisfies the initial condition

$$y(0) = \frac{7}{2}, \quad y'(0) = -4, \quad y''(0) = \frac{5}{2}, \quad y'''(0) = -2\,.$$

The roots of the characteristic equation are $r = \pm 1$ and $r = \pm i$. Therefore, the general solution to the ODE above is

$$y = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t\,.$$

To satisfy the initial condition, $C_1, \cdots, C_4$ has to satisfy

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ -4 \\ \frac{5}{2} \\ -2 \end{bmatrix}.$$

Solving the linear system above, we find that the solution solving the ODE with the given initial data is

$$y = 3e^{-t} + \frac{1}{2}\cos t - \sin t\,.$$

Example 4.10. Find the general solution of

$$y^{(4)} + y = 0\,.$$

The roots of the characteristic equation are $r = \pm\big(\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i\big)$. Therefore, the general solution to the ODE above is

$$y = \exp\big(\frac{\sqrt{2}}{2}t\big)\big(C_1 \cos\frac{\sqrt{2}}{2}t + C_2 \sin\frac{\sqrt{2}}{2}t\big) + \exp\big(-\frac{\sqrt{2}}{2}t\big)\big(C_2 \cos\frac{\sqrt{2}}{2}t + C_4 \sin\frac{\sqrt{2}}{2}t\big)\,.$$

4.3 The Method of Variation of Parameters

To solve a non-homogeneous ODE

$$L[y] = \frac{d^n y}{dt} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)\,, \tag{4.1}$$

often times we apply the method of variation of parameters to find a particular solution. Suppose that $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of the homogeneous equation (4.3), we assume that a particular solution can be written as

$$y = Y(t) = u_1(t)\varphi_1(t) + \cdots + u_n(t)\varphi_n(t)\,.$$

Assume that $u_1, \cdots, u_n$ satisfy

$$u_1'\varphi_1^{(j)} + \cdots + u_n'\varphi_n^{(j)} = 0$$

for $j = 0, \cdots, 1, n-2$. Then

$$\begin{aligned}
Y' &= u_1\varphi_1' + \cdots + u_n\varphi_n'\,,\\
Y'' &= u_1\varphi_1'' + \cdots + u_n\varphi_n''\,,\\
&\ \vdots\\
Y^{(n-1)} &= u_1\varphi_1^{(n-1)} + \cdots + u_n\varphi_n^{(n-1)}\,,
\end{aligned}$$

and

$$Y^{(n)} = u_1'\varphi_1^{(n-1)} + \cdots + u_n'\varphi_n^{(n-1)} + u_1\varphi_1^{(n)} + \cdots + u_n\varphi_n^{(n)}\,.$$

Since $y = Y(t)$ is assumed to be a particular solution of (4.1), we have

$$u_1'\varphi_1^{(n-1)} + \cdots + u_n'\varphi_n^{(n-1)} = g(t)\,.$$

Therefore, $u_1, \cdots, u_n$ satisfy

$$\begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix}.$$

Let W_m denote the Wronskian of $\{\varphi_1, \cdots, \varphi_{m-1}, \varphi_{m+1}, \cdots, \varphi_n\}$; that is,

$$W_m = \begin{vmatrix} \varphi_1 & \cdots & \varphi_{m-1} & \varphi_{m+1} & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_{m-1}' & \varphi_{m+1}' & \cdots & \varphi_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_i^{(n-2)} & \cdots & \varphi_{m-1}^{(n-2)} & \varphi_{m+1}^{(n-2)} & \cdots & \varphi_n^{(n-2)} \end{vmatrix}.$$

Then $u_i' = (-1)^{n+i}\dfrac{W_i}{W(\varphi_1, \cdots, \varphi_n)}$ which implies that

$$Y(t) = \sum_{i=1}^{n} (-1)^{n+i}\varphi_i(t)\int_{t_0}^{t} \frac{W_i(s)g(s)}{W(\varphi_1, \cdots, \varphi_n)(s)}\,ds\,.$$

Example 4.11. Find the general solution to

$$y''' - y'' - y' + y = g(t)\,. \tag{4.8}$$

Note the the roots of the characteristic equation $r^3 - r^2 - r + 1 = 0$ of the homogeneous equation

$$y''' - y'' - y' + y = 0 \tag{4.9}$$

are $r = 1$ (double) and $r = -1$; thus we have a fundamental set $\{e^t, te^t, e^{-t}\}$ of equation (4.9). Let $\varphi_1(t) = e^t$, $\varphi_2(t) = te^t$ and $\varphi_3(t) = e^{-t}$. Then

$$W(\varphi_1, \varphi_2, \varphi_3)(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix} = \big[(t+1) + (t+2) - t - (t+1) - t + (t+2)\big]e^t = 4e^t\,,$$

and $W_1(t) = -2t - 1$, $W_2(t) = -2$ and $W_3(t) = e^{2t}$. Therefore, a particular solution is

$$\begin{aligned} Y(t) &= e^t \int_0^t \frac{(-2s-1)}{4e^s} g(s)ds - te^t \int_0^s \frac{-2}{4e^s} g(s)ds + e^{-t} \int_0^t \frac{e^{2s}}{4e^s} g(s)ds \\ &= \frac{1}{4} \int_0^t \big[2(t-s) - 1)e^{t-s} + e^{s-t}\big] g(s)ds\,, \end{aligned}$$

and the general solution to (4.8) is

$$y = C_1 e^t + C_2 te^t + C_3 e^{-t} + Y(t)\,.$$

5 Series Solutions of Second Order Linear Equations

5.1 Properties of Power Series

Definition 5.1. A power series about c is a series of the form $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ for some sequence $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ (or $\mathbb{C}$) and $c \in \mathbb{R}$ (or $\mathbb{C}$).

Proposition 5.2. *If a power series centered at c is convergent at some point $b \neq c$, then the power series converges absolutely for all points in $(c - |b-c|, c + |b-c|)$.*

Proof. Since the series $\sum\limits_{k=0}^{\infty} a_k(b-c)^k$ converges, $|a_k||b-c|^k \to 0$ as $k \to \infty$; thus there exists $M > 0$ such that $|a_k||b-c|^k \leqslant M$ for all k. Then if $x \in (c - |b-c|, c + |b-c|)$, the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely since

$$\sum_{k=0}^{\infty} |a_k(x-c)^k| = \sum_{k=0}^{\infty} |a_k||x-c|^k = \sum_{k=0}^{\infty} |a_k||b-c|^k \frac{|x-c|^k}{|b-c|^k} \leqslant M \sum_{k=0}^{\infty} \Big(\frac{|x-c|}{|b-c|}\Big)^k$$

which converges (because of the geometric series test or ratio test). ∎

Definition 5.3. A number R is called the ***radius of convergence*** of the power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ if the series converges for all $x \in (c-R, c+R)$ but diverges if $x > c+R$ or $x < c-R$. In other words,

$$R = \sup\Big\{r \geqslant 0 \,\Big|\, \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges in } [c-r, c+r]\Big\}.$$

The ***interval of convergence*** or ***convergence interval*** of a power series is the collection of all x at which the power series converges.

We remark that Proposition 5.2 implies that a power series converges absolutely in the interior of the interval of convergence.

Proposition 5.4. *A power series is continuous in the interior of the convergence interval; that is, if $R > 0$ is the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$, then $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ is continuous in $(c-R, c+R)$.*

Proof. W.L.O.G., we prove that the power series is continuous at $x_0 \in [c, c+R)$. Let $\varepsilon > 0$ be given. Define $r = \dfrac{c+R-x_0}{2}$. Then $|r| < R$; thus there exists $N > 0$ such that

$$\sum_{k=N+1}^{\infty} |a_k| r^k < \frac{\varepsilon}{4}.$$

Moreover, since $\sum\limits_{k=0}^{N} a_k(x-c)^k$ is continuous at x_0, there exists $0 < \delta < r$ such that

$$\Big|\sum_{k=0}^{N} a_k(x-c)^k - \sum_{k=0}^{N} a_k(x_0-c)^k\Big| < \frac{\varepsilon}{2} \qquad \forall\, |x-x_0| < \delta.$$

Therefore, if $|x-x_0| < \delta$, we have

$$\begin{aligned}
&\Big|\sum_{k=0}^{\infty} a_k(x-c)^k - \sum_{k=0}^{\infty} a_k(x_0-c)^k\Big| \\
&\quad \leqslant \Big|\sum_{k=0}^{N} a_k(x-c)^k - \sum_{k=0}^{N} a_k(x_0-c)^k\Big| + \sum_{k=N+1}^{\infty} |a_k| r^k \frac{|x-c|^k}{r^k} + \sum_{k=N+1}^{\infty} |a_k| r^k \frac{|x_0-c|^k}{r^k} \\
&\quad \leqslant \Big|\sum_{k=0}^{N} a_k(x-c)^k - \sum_{k=0}^{N} a_k(x_0-c)^k\Big| + 2\sum_{k=N+1}^{\infty} |a_k| r^k < \varepsilon
\end{aligned}$$

which implies that $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ is continuous at x_0. □

Theorem 5.5. *Let $R > 0$ be the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$. Then*

$$\int_c^x \sum_{k=0}^{\infty} a_k(t-c)^k dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-c)^{k+1} \qquad \forall\, x \in (c-R, c+R).$$

Proof. W.L.O.G., we assume that $x \in (c, c+R)$. Let $\varepsilon > 0$ be given. Choose $x_0 \in (c-R, c+R)$ such that $|x-c| < |x_0 - c|$. Then for $t \in [c, x]$, $\dfrac{|t-c|}{|x_0-c|} \leqslant 1$. Moreover, since $\sum\limits_{k=1}^{\infty} a_k (x_0 - c)^k$ converges absolutely, there exists $N \geqslant 0$ such that

$$\sum_{k=N+1}^{\infty} |a_k||x_0 - c|^k \leqslant \frac{\varepsilon}{|x_0 - c|}\,.$$

Since

$$\begin{aligned}\int_c^x \sum_{k=0}^{\infty} a_k (t-c)^k dt &= \int_c^x \sum_{k=0}^{n} a_k (t-c)^k dt + \int_c^x \sum_{k=N+1}^{\infty} a_k (t-c)^k dt \\ &= \sum_{k=0}^{n} \frac{a_k}{k+1} (x-c)^{k+1} + \int_c^x \sum_{k=n+1}^{\infty} a_k (t-c)^k dt\,,\end{aligned}$$

we have for $n \geqslant N$,

$$\begin{aligned}\Big| \int_c^x \sum_{k=0}^{\infty} a_k (t-c)^k dt - \sum_{k=0}^{n} \frac{a_k}{k+1} (x-c)^{k+1} \Big| &\leqslant \int_c^x \sum_{k=n+1}^{\infty} |a_k||x_0 - c|^k \frac{(t-c)^k}{|x_0-c|^k} dt \\ \leqslant \int_c^x \sum_{k=N+1}^{\infty} |a_k||x_0 - c|^k dt &\leqslant |x_0 - c| \sum_{k=N+1}^{\infty} |a_k||x_0-c|^k < \varepsilon\,.\end{aligned}$$

In other words, $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{a_k}{k+1} (x-c)^{k+1} = \displaystyle\int_c^x \sum_{k=0}^{\infty} a_k (t-c)^k dt$ which concludes the corollary. □

Theorem 5.6. *Let $R > 0$ be the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k (x-c)^k$. Then*

$$\frac{d}{dx} \sum_{k=0}^{\infty} a_k (x-c)^k = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1} \qquad \forall\, x \in (c-R, c+R)\,.$$

Proof. We first show that the series $\sum\limits_{k=1}^{\infty} k a_k (x-c)^{k-1}$ also converges for all $x \in (c-R, c+R)$. Let $x \in (c-R, c+R)$. Then there exists $x_0 \in (c-R, c+R)$ such that $|x-c| < |x_0 - c|$. Choose $N > 0$ such that

$$k \frac{|x-c|^k}{|x_0-c|^k} \leqslant 1 \qquad \text{if} \quad k \geqslant N\,.$$

We note that it is possible to find such an N since $\lim\limits_{k\to\infty} k \dfrac{|x-c|^k}{|x_0-c|^k} = 0$. Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty} k|a_k||x-c|^k &= \sum_{k=0}^{N} k|a_k||x-c|^k + \sum_{k=N+1}^{\infty} k|a_k||x-c|^k \\ &\leqslant \sum_{k=0}^{N} k|a_k||x-c|^k + \sum_{k=N+1}^{\infty} |a_k||x_0-c|^k k \frac{|x-c|^k}{|x_0-c|^k} \\ &\leqslant \sum_{k=0}^{N} k|a_k||x-c|^k + \sum_{k=N+1}^{\infty} |a_k||x_0-c|^k < \infty\end{aligned}$$

which implies that the series $\sum_{k=0}^{\infty} k|a_k||x-c|^k$ converges absolutely.

Now, Theorem 5.5 implies that

$$\int_c^x \sum_{k=1}^{\infty} k a_k (t-c)^{k-1} dt = \int_c^x \sum_{k=0}^{\infty} (k+1) a_{k+1} (t-c)^k dt = \sum_{k=0}^{\infty} a_{k+1} (x-c)^{k+1} = \sum_{k=1}^{\infty} a_k (x-c)^k ;$$

thus we have

$$a_0 + \int_c^x \sum_{k=1}^{\infty} k a_k (t-c)^{k-1} dt = \sum_{k=0}^{\infty} a_k (x-c)^k .$$

Moreover, Proposition 5.4 implies that the power series $\sum_{k=0}^{\infty} k|a_k||x-c|^k$ is continuous in $(c-R, c+R)$. As a consequence, the fundamental theorem of Calculus implies that

$$\sum_{k=1}^{\infty} k a_k (x-c)^{k-1} = \frac{d}{dx} \int_c^x \sum_{k=1}^{\infty} k a_k (t-c)^{k-1} dt = \frac{d}{dx} \sum_{k=0}^{\infty} a_k (x-c)^k$$

which concludes the theorem. □

Definition 5.7. A function $f : (a,b) \to \mathbb{R}$ is said to be ***analytic*** at $c \in (a,b)$ if f is infinitely many times differentiable at c and there exists $R > 0$ such that

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \qquad \forall\, x \in (c-R, c+R) \subseteq (a,b)$$

for some sequence $\{a_k\}_{k=0}^{\infty}$.

Remark 5.8. If $f : (a,b) \to \mathbb{R}$ is analytic at $c \in (a,b)$, then Theorem 5.6 implies that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \qquad \forall\, x \in (c-R, c+R) \subseteq (a,b)$$

for some $R > 0$.

A function which is infinitely many times differentiable at a point c might not be analytic at c. For example, consider the function

$$f(x) = \begin{cases} \exp\big(-\dfrac{1}{x^2}\big) & \text{if } x \neq 0 \,, \\ 0 & \text{if } x = 0 \,. \end{cases}$$

Then $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$ which implies that f cannot be analytic at 0.

5.1.1 Product of Power Series

Definition 5.9. Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, the series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N} \cup \{0\}$, is called the ***Cauchy product*** of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

Theorem 5.10. *Suppose that the two series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ converge absolutely. Then the Cauchy product of $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ converges absolutely to $\Big(\sum\limits_{n=0}^{\infty} a_n\Big)\Big(\sum\limits_{n=0}^{\infty} b_n\Big)$; that is,*

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big) = \Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big).$$

Proof. **Claim**: If $\sum\limits_{n=0}^{\infty} a_n$ converges absolutely and $\pi : \mathbb{N} \to \mathbb{N}$ is bijective (that is, one-to-one and onto), then $\sum\limits_{n=0}^{\infty} a_{\pi(n)}$ converges absolutely to $\sum\limits_{n=0}^{\infty} a_n$.

Proof of claim: Let $\sum\limits_{n=0}^{\infty} a_n = a$ and $\varepsilon > 0$ be given. Since $\sum\limits_{n=0}^{\infty} a_n$ converges absolutely, there exists $N > 0$ such that

$$\sum_{n=N+1}^{\infty} |a_n| < \frac{\varepsilon}{2}.$$

Let $K = \max\big\{\pi^{-1}(1), \cdots, \pi^{-1}(N)\big\} + 1$. Then if $k \geqslant K$, $\pi(k) \geqslant N+1$; thus if $k \geqslant K$,

$$\sum_{n=k+1}^{\infty} |a_{\pi(n)}| \leqslant \sum_{n=N+1}^{\infty} |a_n| < \frac{\varepsilon}{2}$$

and

$$\Big|\sum_{n=0}^{k} a_{\pi(n)} - a\Big| \leqslant \Big|\sum_{n=0}^{k} a_{\pi(n)} - \sum_{n=0}^{N} a_n\Big| + \Big|\sum_{n=0}^{N} a_n - a\Big| \leqslant 2\sum_{n=N+1}^{\infty} |a_n| < \varepsilon.$$

Therefore, $\sum\limits_{n=0}^{\infty} a_{\pi(n)}$ converges absolutely to a.

Claim: If $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ converge absolutely, then $\sum\limits_{n,m=1}^{\infty} a_n b_m$ converges absolutely and

$$\sum_{n,m=1}^{\infty} a_n b_m = \Big(\sum_{n=1}^{\infty} a_n\Big)\Big(\sum_{m=1}^{\infty} b_m\Big),$$

where $\sum\limits_{n,m=1}^{\infty} a_n b_m$ denotes the limit $\lim\limits_{N,M\to\infty} \sum\limits_{n=1}^{N}\sum\limits_{m=1}^{M} a_n b_m$.

Proof of claim: If $N_1 < N_2$ and $M_1 < M_2$,

$$\Big|\Big(\sum_{n=1}^{N_1} a_n\Big)\Big(\sum_{m=0}^{M_1} b_m\Big) - \Big(\sum_{n=1}^{N_2} a_n\Big)\Big(\sum_{m=0}^{M_2} b_m\Big)\Big| \leqslant \sum_{n=1}^{N_1} |a_n| \sum_{m=M_1+1}^{M_2} |b_m| + \sum_{n=N_1+1}^{N_2} |a_n| \sum_{m=1}^{M_2} |b_m|$$

thus

$$\begin{aligned}
&\Big|\Big(\sum_{n=1}^{N_1} a_n\Big)\Big(\sum_{m=0}^{M_1} b_m\Big) - \sum_{n,m=1}^{\infty} a_n b_m\Big| \\
&\quad = \lim_{N_2,M_2\to\infty}\Big|\Big(\sum_{n=1}^{N_1} a_n\Big)\Big(\sum_{m=0}^{M_1} b_m\Big) - \Big(\sum_{n=1}^{N_2} a_n\Big)\Big(\sum_{m=0}^{M_2} b_m\Big)\Big| \\
&\quad \leqslant \Big(\sum_{n=1}^{\infty} |a_n| + \sum_{m=1}^{\infty} |b_m|\Big)\Big(\sum_{n=N_1+1}^{\infty} |a_n| + \sum_{m=M_1+1}^{\infty} |b_m|\Big).
\end{aligned}$$

The claim is then concluded by passing to the limit as $M_1 \to \infty$ and then $N_1 \to \infty$.

The theorem follows from the fact that the Cauchy product is a special rearrangement of the series $\sum_{n,m=1}^{\infty} a_n b_m$. □

Corollary 5.11. *Let $R_1, R_2 > 0$ be the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ and $\sum_{k=0}^{\infty} b_k(x-c)^k$, respectively. Then with R denoting $\min\{R_1, R_2\}$, we have*

$$\Big(\sum_{k=0}^{\infty} a_k(x-c)^k\Big)\Big(\sum_{k=0}^{\infty} b_k(x-c)^k\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)(x-c)^n \qquad \forall\, x \in (c-R, c+R)\,.$$

5.1.2 General Theory

The discussion of the power series is for the purpose of solving ODE with analytic coefficients and forcings.

Theorem 5.12 (Cauchy-Kowalevski, Special case)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and $f : \Omega \times (t_0 - h, t_0 + h) \to \mathbb{R}^n$ be an analytic function in some neighborhood (x_0, t_0) for some $x_0 \in \Omega$; that is, for some $r > 0$,*

$$f(y,t) = f(y_0,t_0) + \sum_{k=1}^{\infty}\sum_{|\alpha|+j=k} c_{\alpha,j}(y-y_0)^\alpha (t-t_0)^j \qquad \forall\,(y,t) \in B\big((y_0,t_0), r\big)\,,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index satisfying $y^\alpha = y_1^{\alpha_1}\cdots y_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots \alpha_n$. Then there exists $0 < \delta < h$ such that the ODE $y'(t) = f(y,t)$ with initial condition $y(t_0) = y_0$ has a unique analytic solution in the interval $(t_0 - \delta, t_0 + \delta)$.

Remark 5.13. If f is continuous at $(y_0 - k, y_0 + k) \times (t_0 - h, t_0 + h)$, then the general existence and uniqueness theorem guarantees the existence of a unique solution of $y'(t) = f(y,t)$ with initial condition $y(t_0) = y_0$ in some time interval $(t_0 - \delta, t_0 + \delta)$. Theorem 5.12 further implies that the solution is analytic if the "forcing" function f is analytic.

5.2 Series Solutions Near an Ordinary Point: Part I

In the remaining chapter we focus on the second order linear homogeneous ODE

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0\,, \tag{5.1}$$

where P, Q, R are assumed to have no common factors. **We note that we change the independent variable from t to x.**

Definition 5.14. A point x_0 is said to be a ***ordinary point*** to ODE (5.1) if $P(x_0) \neq 0$, and the two functions Q/P, R/P are analytic at x_0. It is called a ***singluar point*** if it is not a regular point. It is called a ***regular singular point*** if the two limits

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$$

both exist and are finite. Any singular point that is not a regular singular point is called an ***irregular singular point***.

If x_0 is a regular point to ODE (5.1), then

$$y'' + p(x)y' + q(x)y = 0$$

for some function p and q that are analytic at x_0. Write $y' = z$. Then the vector $\boldsymbol{w} = (y, z)$ satisfies

$$\boldsymbol{w}' = \frac{d}{dx}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -p(x)z - q(x)y \end{bmatrix} \equiv f(x, \boldsymbol{w}).$$

It is clear that f is analytic at x_0 if p, q are analytic at x_0; thus the Cauchy-Kowalevski theorem implies that there exists a unique analytic solution.

Example 5.15. Find a series solution to $y'' + y = 0$.

Suppose that the solution can be written as $y = \sum\limits_{k=0}^{\infty} a_k x^k$. Then Theorem 5.6 implies that

$$y'' = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k;$$

thus $y'' + y = 0$ implies that

$$\sum_{k=0}^{\infty}\big[(k+2)(k+1)a_{k+2} + a_k\big]x^k = 0.$$

Since the power series (representation) associated to the zero function is $\sum\limits_{k=0}^{\infty} 0 \cdot x^k$, we must have $a_{k+2} = \dfrac{-a_k}{(k+2)(k+1)}$ for all $k \in \mathbb{N} \cup \{0\}$; thus we conclude that

$$a_{2k} = \frac{-a_{2k-2}}{(2k)(2k-1)} = \frac{a_{2k-4}}{(2k)(2k-1)(2k-2)(2k-3)} = \cdots = \frac{(-1)^k a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{-a_{2k-1}}{(2k+1)(2k)} = \frac{a_{2k+1}}{(2k)(2k-1)(2k-2)(2k-3)} = \cdots = \frac{(-1)^k a_1}{(2k+1)!}.$$

Therefore,

$$y = \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} = a_0\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} + a_1\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}.$$

Let $C(x) = \sum\limits_{k=0}^{\infty} a_{2k}x^{2k}$ and $S(x) = \sum\limits_{k=0}^{\infty} a_{2k+1}x^{2k+1}$. Then it is clear that $C'(x) = -S(x)$ and $S'(x) = C(x)$. Moreover, the Wronskian of $\{C, S\}$ at $x = 0$ is

$$\begin{vmatrix} C(0) & S(0) \\ C'(0) & S'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

which implies that $\{C, S\}$ is a fundamental set of equation $y'' + y = 0$.

Example 5.16. Find a series solution to Airy's equation $y'' - xy = 0$.

Suppose that the solution can be written as $y = \sum\limits_{k=0}^{\infty} a_k x^k$. Then

$$y'' = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k\,;$$

and

$$xy = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k\,.$$

Therefore,

$$a_2 + \sum_{k=1}^{\infty}\big[(k+2)(k+1)a_{k+2} - a_{k-1}\big]x^k = 0$$

which implies that $a_2 = 0$ and $a_{k+2} = \dfrac{a_{k-1}}{(k+2)(k+1)}$ for all $k \in \mathbb{N}$. The recurrence relation further implies that $a_5 = a_8 = a_{11} = \cdots = a_{3k-1} = \cdots = 0$ for all $k \in \mathbb{N}$. Furthermore, we have

$$\begin{aligned} a_{3k} &= \frac{a_{3k-3}}{(3k)(3k-1)} = \frac{a_{3k-6}}{(3k)(3k-1)(3k-3)(3k-4)} = \cdots \\ &= \frac{a_0}{(3k)(3k-1)(3k-3)(3k-4)\cdots 3\cdot 2} = \frac{(3k-2)(3k-5)\cdots 4\cdot 1a_0}{(3k)!} \\ &= \frac{3^k\big(k-\frac{2}{3}\big)\big(k-\frac{5}{3}\big)\cdots\frac{1}{3}a_0}{(3k)!} = \frac{3^k\Gamma(k+1/3)}{\Gamma(1/3)(3k)!}a_0 \end{aligned}$$

and

$$\begin{aligned} a_{3k+1} &= \frac{a_{3k-2}}{(3k+1)(3k)} = \frac{a_{3k-5}}{(3k+1)(3k)(3k-2)(3k-3)} = \cdots \\ &= \frac{a_1}{(3k+1)(3k)(3k-2)(3k-3)\cdots 4\cdot 3} = \frac{(3k-1)(3k-4)\cdots 2a_1}{(3k+1)!} \\ &= \frac{3^k\big(k-\frac{1}{3}\big)\big(k-\frac{4}{3}\big)\cdots\frac{2}{3}a_1}{(3k+1)!} = \frac{3^k\Gamma(k+2/3)}{\Gamma(2/3)(3k+1)!}a_1\,. \end{aligned}$$

Therefore, the solution of Airy's equation is of the form

$$y = a_0\sum_{k=0}^{\infty}\frac{3^k\Gamma(k+1/3)}{\Gamma(1/3)(3k)!}x^{3k} + a_1\sum_{k=0}^{\infty}\frac{3^k\Gamma(k+2/3)}{\Gamma(2/3)(3k+1)!}x^{3k+1}\,.$$

Example 5.17. In this example, instead of considering a series solution of Airy's equation $y''-xy = 0$ of the form $y = \sum\limits_{k=0}^{\infty} a_k x^k$, we look for a solution of the form $y = \sum\limits_{k=0}^{\infty} a_k(x-1)^k$.

Since

$$y'' = \sum_{k=2}^{\infty} k(k-1)a_k(x-1)^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^k$$

and

$$xy = (x-1)y + y = \sum_{k=0}^{\infty} a_k(x-1)^{k+1} + \sum_{k=0}^{\infty} a_k(x-1)^k = \sum_{k=1}^{\infty} a_{k-1}(x-1)^k + \sum_{k=0}^{\infty} a_k(x-1)^k\,,$$

we have

$$(2a_2 - a_0) + \big[6a_3 - (a_1 + a_0)\big](x-1) + \sum_{k=2}^{\infty}\big[(k+2)(k+1)a_{k+2} - (a_{k-1} + a_k)\big](x-1)^k = 0\,.$$

Therefore, $2a_2 = a_0$, $6a_3 = a_1 + a_0$, $12a_4 = a_2 + a_1$, $20a_5 = a_3 + a_2$, and in general,

$$(k+2)(k+1)a_{k+2} = a_{k+1} + a_k\,.$$

Solving for a few terms, we find that

$$a_2 = \frac{1}{2}a_0\,, \quad a_3 = \frac{1}{6}a_0 + \frac{1}{6}a_1\,, \quad a_4 = \frac{1}{24}a_0 + \frac{1}{12}a_1\,, \quad a_5 = \frac{1}{30}a_0 + \frac{1}{120}a_1\,, \quad \cdots$$

It seems not possible to find a general form the the series solution. Nevertheless, we have

$$\begin{aligned} y &= a_0\Big[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots\Big] \\ &\quad + a_1\Big[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots\Big]\,. \end{aligned}$$

5.3 Series Solution Near an Ordinary Point: Part II

There is another way to computed the coefficients a_k of the series solution to ODE (5.1). The idea is to differentiate the equation (5.1) k-times and then evaluate at an ordinary point x_0 so that $y^{k+2}(x_0)$ can be obtained once $y^j(x_0)$'s are known for $0 \leqslant j \leqslant k+1$. To be more precise, we differentiate (5.1) k-times and use the Leibniz rule to obtain that

$$P(x_0)y^{(k+2)}(x_0) + \sum_{j=0}^{k-1} C^k_j P^{(k-j)}(x_0)y^{(j+2)}(x_0) + \sum_{j=0}^{k} C^k_j\big(Q^{(k-j)}(x_0)y^{(1+j)}(x_0) + R^{(k-j)}(x_0)y^{(j)}(x_0)\big) = 0\,;$$

thus

$$\begin{aligned} P(x_0)y^{(k+2)}(x_0) &= -\sum_{j=2}^{k+1} C^k_{j-2}P^{(k-j+2)}(x_0)y^{(j)}(x_0) - \sum_{j=1}^{k+1} C^k_{j-1}Q^{(k-j+1)}(x_0)y^{(j)}(x_0) \\ &\quad - \sum_{j=0}^{k} C^k_j R^{(k-j)}(x_0)y^{(j)}(x_0) \\ &= -\big[kP'(x_0) + Q(x_0)\big]y^{(k+1)}(x_0) - \big[Q^{(k)}(x_0) + kR^{(k-1)}(x_0)\big]y'(x_0) - R^{(k)}(x_0)y(x_0) \\ &\quad - \sum_{j=2}^{k}\big[C^k_{j-2}P^{(k-j+2)}(x_0) + C^k_{j-1}Q^{(k-j+1)}(x_0) + C^k_j R^{(k-j)}(x_0)\big]y^{(j)}(x_0)\,. \end{aligned}$$

The recurrence relation above can be used to obtain the coefficients $a_{k+2} = \dfrac{y^{(k+2)}(x_0)}{(k+2)!}$ of the series solution $y = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ to (5.1) once $y^{k+1}(x_0), \cdots, f(x_0)$ are known.

Example 5.18. Find the series solution about 1 of Airy's equation $y'' - xy = 0$.

Assume that the series solution is $y = \sum\limits_{k=0}^{\infty} a_k(x-1)^k$. First, we know that $y''(1) - y(1) = 0$. Since $y(1) = a_0$, we know that $a_2 = \dfrac{y''(1)}{2} = \dfrac{a_0}{2}$. Differentiating Airy's equation k-times, we find that

$$y^{(k+2)} - xy^{(k)} - ky^{(k-1)} = 0\,;$$

thus

$$(k+2)!a_{k+2} = y^{(k+2)}(1) = y^{(k)}(1) + ky^{(k-1)} = k!a_k + k(k-1)!a_{k-1} = k!(a_k + a_{k-1})\,.$$

Therefore, $(k+2)(k+1)a_{k+2} = a_k + a_{k-1}$ which is exactly what we use to obtain the series solution about 1 to Airy's equation.

Theorem 5.19. *Let x_0 be an ordinary point of ODE* (5.1)*, and R_1 and R_2 are the radius of convergence of the power series representation of $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ about x_0. Suppose that $y = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ is the unique analytic solution to* (5.1) *with initial condition $y(x_0) = a_0$ and $y'(x_0) = a_1$. Then the radius of convergence of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ is the at least as large as* $\min\{R_1, R_2\}$.

Example 5.20. The radius of convergence of series solutions about any point $x = x_0$ of the ODE

$$y'' + (\sin x)y' + (1+x^2)y = 0$$

is infinite; that is, for any $x_0 \in \mathbb{R}$, series solutions about $x = x_0$ of the ODE above converge for all $x \in \mathbb{R}$.

Example 5.21. Find a lower bound for the radius of convergence of series solutions about $x = 0$ of the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0\,.$$

Since the Taylor series expansion of $\frac{1}{1-x^2}$ about 0 converges for all $x \in (-1,1)$, the power series representation of $\frac{2x}{1-x^2}$ about 0 converges for all $x \in (-1,1)$; thus the radius of convergence of $\frac{2x}{1-x^2}$ is 1. Therefore, the radius of convergence of the series solution about 0 of the Legendre equation is at least 1. We also note that ± 1 are both regular singular point of the Legendre equation.

Example 5.22. Find a lower bound for the radius of convergence of series solutions about $x = 0$ or about $x = -\frac{1}{2}$ of the ODE

$$(1+x^2)y'' + 2xy' + 4x^2y = 0\,.$$

Similar to the previous example, since the Taylor series expansion of $\frac{1}{1+x^2}$ about 0 converges for all $x \in (-1,1)$, the radius of convergence of the series solution of the ODE is at least 1.

Next, consider the series solution about $-\frac{1}{2}$. Since both x and x^2 are polynomials, it suffices to find the radius of convergence of the power series representation of $\frac{1}{1+x^2}$ about $-\frac{1}{2}$. Nevertheless, the radius of convergence of the power series representation of $\frac{1}{1+x^2}$ about $-\frac{1}{2}$ is $\frac{\sqrt{5}}{2}$.

5.4 Euler Equations; Regular Singular Points

In this section we consider the Euler equation

$$x^2y'' + \alpha xy' + \beta y = 0\,. \tag{5.2}$$

Note that $x_0 = 0$ is a regular singular point of (5.2).

Assume that we only consider the solution of the Euler equation in the region $x > 0$. Let $z(t) = y(e^t)$. Then $z'(t) = y'(e^t)e^t$ and $z''(t) = y''(e^t)e^{2t} + y'(e^t)e^t$ which implies that $y''(e^t)e^{2t} = z''(t) - z'(t)$. Therefore,

$$z''(t) + (\alpha - 1)z'(t) + \beta z(t) = 0\,. \tag{5.3}$$

This is a second order ODE with constant coefficients, and can be solved by looking at the multiplicity and complexity of the roots of the characteristic equation

$$r^2 + (\alpha - 1)r + \beta = 0\,. \tag{5.4}$$

We note that (5.4) can also be written as $r(r-1)+\alpha r+\beta = 0$, and is called the ***indicial equation***.

1. Suppose the roots of the characteristic equation are distinct real numbers r_1 and r_2. Then the solution to (5.3) is $z(t) = C_1e^{r_1t} + C_2e^{r_2t}$; thus the solution to the Euler equation is
$$y(x) = C_1e^{r_1\log x} + C_2e^{r_2\log x} = C_1x^{r_1} + C_2x^{r_2}\,.$$

2. Suppose the characteristic equation has a real double root r. Then the solution to (5.3) is $z(t) = (C_1t + C_2)e^{rt}$; thus the solution to the Euler equation is
$$y(x) = (C_1\log x + C_2)e^{r\log x} = (C_1\log x + C_2)x^r\,.$$

3. Suppose the roots of the characteristic equation are complex numbers $r_1 = a+bi$ and $r_2 = a-bi$. Then the solution to (5.3) is $z(t) = C_1e^{at}\cos(bt) + C_2e^{at}\sin(bt)$; thus the solution to the Euler equation is
$$y(x) = C_1e^{a\log x}\cos(b\log x) + C_1e^{a\log x}\sin(b\log x) = C_1x^a\cos(b\log x) + C_2x^a\sin(b\log x)\,.$$

Now we consider the solution to (5.2) in the region $x < 0$. We then let $z(x) = y(-x)$ and find that z satisfies also satisfies the same Euler equation; that is,

$$x^2z'' + \alpha xz' + \beta z = 0\,.$$

We can then solve for z by looking at the multiplicity and complexity of the roots of the characteristic equation, and conclude that

1. Case 1 - Distinct real roots r_1 and r_2:
$$y(x) = C_1|x|^{r_1} + C_2|x|^{r_2}\,.$$

2. Case 2 - Double real root r:
$$y(x) = (C_1|x| + C_2)|x|^r\,.$$

3. Case 3 - Complex roots $a \pm bi$:
$$y(x) = C_1|x|^{at}\cos(b\log|x|) + C_2|x|^{at}\sin(b\log|x|)\,.$$

5.5 Series Solutions Near a Regular Singular Point: Part I

Suppose that x_0 is a regular singular point of (5.1); that is, $P(x_0) = 0$, and both limits

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$$

exist. W.L.O.G., we can assume that $x_0 = 0$ (otherwise make a change of variable $\widetilde{x} = x - x_0$), and only focus the discussion of the solution in the region $x > 0$. Suppose that

$$x\frac{Q(x)}{P(x)} = \sum_{k=0}^{\infty} p_k x^k \quad \text{and} \quad x^2\frac{R(x)}{P(x)} = \sum_{k=0}^{\infty} q_k x^k$$

in some interval $(-R, R)$. Then by multiplying both side of (5.1) by $\dfrac{x^2}{P(x)}$, we obtain that

$$x^2 y'' + x\Big(\sum_{k=0}^{\infty} p_k x^k\Big)y' + \Big(\sum_{k=0}^{\infty} q_k x^k\Big)y = 0\,. \tag{5.5}$$

We note that if $p_k = q_k = 0$ for all $k \in \mathbb{N}$, the equation above is the Euler equation

$$x^2 y'' + p_0 x y' + q_0 y = 0 \tag{5.6}$$

that we discussed in previous section.

For x near 0, it is "reasonable" to expect that the solution to (5.5) will behave like the solution to the Euler equation

$$x^2 y'' + p_0 x y' + q_0 y = 0\,.$$

The idea (due to Frobenius) of solving (5.5) is that the solution of (5.5) should be of the form x^r times an analytic function. Hence we look for solutions of (5.5) of the form

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}\,, \qquad x > 0\,, \tag{5.7}$$

where a_0 is assumed to be non-zero (otherwise we replace r by $1 + r$ if $a_1 \neq 0$). Since

$$y'(x) = r x^{r-1} \sum_{k=0}^{\infty} a_k x^k + x^r \sum_{k=0}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1}$$

and accordingly,

$$y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}\,,$$

we obtain

$$\sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r} + \Big(\sum_{k=0}^{\infty} p_k x^k\Big)\Big(\sum_{k=0}^{\infty} (k+r) a_k x^{k+r}\Big) + \Big(\sum_{k=0}^{\infty} q_k x^k\Big)\Big(\sum_{k=0}^{\infty} a_k x^{k+r}\Big) = 0\,,$$

or cancelling x^r,

$$\sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^k + \Big(\sum_{k=0}^{\infty} p_k x^k\Big)\Big(\sum_{k=0}^{\infty} (k+r) a_k x^k\Big) + \Big(\sum_{k=0}^{\infty} q_k x^k\Big)\Big(\sum_{k=0}^{\infty} a_k x^k\Big) = 0\,.$$

Using the Cauchy product, we further conclude that

$$\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^k + \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}(j+r)a_j p_{k-j}\Big)x^k + \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} q_{k-j}a_j\Big)x^k = 0\,.$$

Therefore, we obtain the following recurrence relation:

$$(k+r)(k+r-1)a_k + \sum_{j=0}^{k}(j+r)a_j p_{k-j} + \sum_{j=0}^{k} q_{k-j}a_j = 0 \qquad \forall\, k \in \mathbb{N}\cup\{0\}\,. \tag{5.8}$$

Therefore, with F denoting the function $F(r) = r(r-1) + rp_0 + q_0$, we have

$$F(r+k)a_k + \sum_{j=0}^{k-1}\big[(j+r)p_{k-j} + q_{k-j}\big]a_j = 0 \qquad \forall\, k \in \mathbb{N}\,. \tag{5.9}$$

The case $k = 0$ induces the following

Definition 5.23. If x_0 is a regular singular point of (5.1), then the ***indicial equation*** for the regular singular point x_0 is

$$r(r-1) + p_0 r + q_0 = 0\,, \tag{5.10}$$

where $p_0 = \lim\limits_{x\to x_0}(x-x_0)\dfrac{Q(x)}{P(x)}$ and $q_0 = \lim\limits_{x\to x_0}(x-x_0)^2\dfrac{R(x)}{P(x)}$. The roots of the indicial equation are called the ***exponents*** (***indices***) of the singularity x_0.

Now assume that r_1, r_2 are roots of the indicial equations for a regular singular point x_0.

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 > r_2$. Since F only has two roots, $F(k+r) \neq 0$ for all $k \in \mathbb{N}$. Therefore, for $r = r_1$, (5.9) indeed is a recurrence relation which implies that a_k depends on $a_0, \cdots, a_{k-1}$ and this, in principle, provides a series solution

$$y_1(x) = x^{r_1}\Big[1 + \sum_{k=1}^{\infty}\frac{a_k(r_1)}{a_0}x^k\Big] \tag{5.11}$$

   to (5.5), in which $a_k(r_1)$ denotes the coefficients when $r = r_1$.

   (a) If in addition $r_2 \neq r_1$ and $r_1 - r_2 \notin \mathbb{N}$, the $F(k+r_2) \neq 0$ for all $k \in \mathbb{N}$; thus for $r = r_2$, (5.9) is also a recurrence relation, and this provides another series solution

$$y_2(x) = x^{r_2}\Big[1 + \sum_{k=1}^{\infty}\frac{a_k(r_2)}{a_0}x^k\Big]\,. \tag{5.12}$$

   (b) If $r_1 = r_2$ or $r_1 - r_2 \in \mathbb{N}$, we will discuss later in the next section.

2. If r_1, r_2 are complex roots, then $r_1 - r_2 \notin \mathbb{N}$ and $F(k+r) \neq 0$ for all $k \in \mathbb{N}$ for $r = r_1, r_2$. Letting

$$x^{a+bi} = x^a \cdot x^{bi} = x^a e^{ib\log x} = x^a\big[\cos(b\log x) + i\sin(b\log x)\big]\,,$$

   then (5.11) and (5.12) provide two solutions of (5.5).

Example 5.24. Solve the differential equation

$$2x^2y'' - xy' + (1+x)y = 0\,. \tag{5.13}$$

We note that 0 is a regular singular point of the ODE above; thus we look for a series solution to the ODE above of the form

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k\,.$$

Then r satisfies the indicial equation for 0

$$2r(r-1) - r + 1 = 0$$

which implies that $r = 1$ or $r = \frac{1}{2}$. Since

$$y'(x) = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1} \quad \text{and} \quad y''(x) = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}\,,$$

we obtain that

$$\sum_{k=0}^{\infty}\big[2(k+r)(k+r-1) - (k+r) + 1\big]a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+1} = 0$$

or cancelling x^r,

$$\sum_{k=0}^{\infty}\big[2(k+r)(k+r-1) - (k+r) + 1\big]a_k x^{k} + \sum_{k=1}^{\infty} a_{k-1} x^{k} = 0\,.$$

Therefore,

$$a_k = -\frac{a_{k-1}}{2(k+r)(k+r-1) - (k+r) + 1} \qquad \forall\, k \in \mathbb{N}\,.$$

1. $r = 1$: $a_k = -\dfrac{a_{k-1}}{k(2k+1)}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} a_k &= -\frac{a_{k-1}}{k(2k+1)} = \frac{a_{k-2}}{k(k-1)(2k+1)(2k-1)} = -\frac{a_{k-3}}{k(k-1)(k-2)(2k+1)(2k-1)(2k-3)} \\ &= \frac{(-1)^k}{k!(2k+1)(2k-1)\cdots 1}a_0 = \frac{(2k)(2k-2)(2k-4)\cdots 2(-1)^k}{k!(2k+1)!}a_0 = \frac{(-1)^k 2^k}{(2k+1)!}a_0\,. \end{aligned}$$

This provides a series solution $y_1(x) = \sum_{k=0}^{\infty} \dfrac{(-1)^k 2^k}{(2k+1)!} x^{k+1}$ whose radius of convergence is ∞.

2. $r = \frac{1}{2}$: $a_k = -\dfrac{a_{k-1}}{k(2k-1)}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} a_k &= -\frac{a_{k-1}}{k(2k-1)} = \frac{a_{k-2}}{k(k-1)(2k-1)(2k-3)} = -\frac{a_{k-3}}{k(k-1)(k-2)(2k-1)(2k-3)(2k-5)} \\ &= \frac{(-1)^k}{k!(2k-1)(2k-3)\cdots 1}a_0 = \frac{(-1)^k(2k)(2k-2)\cdots 2}{k!(2k)!}a_0 = \frac{(-1)^k 2^k}{(2k)!}a_0\,. \end{aligned}$$

This provides a series solution $y_2(x) = \sum_{k=0}^{\infty} \dfrac{(-1)^k 2^k}{(2k)!} x^{k+\frac{1}{2}}$ whose radius of convergence is ∞.

Therefore, the general solution to (5.13) in the series form is $y = C_1 y_1(x) + C_2 y_2(x)$.

Example 5.25. Find a series solution about the regular singular point $x = 0$ of

$$(x+2)x^2 y''(x) - xy'(x) + (1+x)y(x) = 0\,, \qquad x > 0\,.$$

Let $p(x) = -\dfrac{1}{x+2}$ and $q(x) = \dfrac{1+x}{x+2}$. Then

$$p(x) = -\frac{1}{2}\frac{1}{1-\frac{-x}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-x)^k}{2^k} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^k}{2^{k+1}}\,,$$

$$q(x) = \frac{x+1}{x+2} = 1 - \frac{1}{2}\frac{1}{1-\frac{-x}{2}} = 1 - \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{2^{k+1}} = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{2^{k+1}}\,.$$

Therefore, $(p_0, q_0) = \left(-\frac{1}{2}, \frac{1}{2}\right)$, and $p_k = q_k = \dfrac{(-1)^{k+1}}{2^{k+1}}$ for all $k \in \mathbb{N}$. The indicial equation for 0 is

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0$$

which implies that $r = 1$ or $r = \dfrac{1}{2}$.

1. $r = 1$: Suppose the series solution to the ODE is $y = x\sum\limits_{k=0}^{\infty} a_k x^k = \sum\limits_{k=0}^{\infty} a_k x^{k+1}$. Then

$$(x+2)x^2\sum_{k=0}^{\infty}(k+1)ka_k x^{k-1} - x\sum_{k=0}^{\infty}(k+1)a_k x^k + (1+x)\sum_{k=0}^{\infty}a_k x^{k+1} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty}(k^2+k+1)a_k x^{k+2} + \sum_{k=0}^{\infty}(2k^2+k)a_k x^{k+1} = 0$$

$$\Rightarrow \sum_{k=1}^{\infty}\Big([(k-1)^2+(k-1)+1]a_{k-1} + (2k^2+k)a_k\Big)x^{k+1} = 0\,.$$

Therefore, $a_k = -\dfrac{k^2-k+1}{(2k+1)k}\,a_{k-1}$ for all $k \in \mathbb{N}$. Note that

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k-1}}\right| = \lim_{k\to\infty}\left|\frac{k^2-k+1}{k(2k+1)}\right| = \frac{1}{2}\,;$$

thus the radius of convergence of the series solution $y = \sum\limits_{k=0}^{\infty} a_k x^{k+1}$ is 2.

2. $r = \dfrac{1}{2}$: Suppose the series solution to the ODE is $y = x^{\frac{1}{2}}\sum\limits_{k=0}^{\infty} a_k x^k = \sum\limits_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}$. Then

$$(x+2)\sum_{k=0}^{\infty}\left(k+\tfrac{1}{2}\right)\left(k-\tfrac{1}{2}\right)a_k x^{k+\frac{1}{2}} - \sum_{k=0}^{\infty}\left(k+\tfrac{1}{2}\right)a_k x^{k+\frac{1}{2}} + \sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}} + \sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty}\left(k^2+\frac{3}{4}\right)a_k x^{k+\frac{3}{2}} + \sum_{k=0}^{\infty}(2k^2-k)a_k x^{k+\frac{1}{2}} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty}\left(\left((k-1)^2+\frac{3}{4}\right)a_{k-1} + (2k^2-k)a_k\right)x^{k+\frac{1}{2}} = 0\,.$$

Therefore, $a_k = -\dfrac{(k-1)^2+3/4}{k(2k-1)}\,a_{k-1}$ for all $k \in \mathbb{N}$. The radius of convergence of this series solution is also 2.

5.6 Series Solutions Near a Regular Singular Point: Part II

5.6.1 The case that the difference of roots of indicial equation is an integer

Suppose that r_1 and r_2 are the roots of the indicial equation for a regular singular point.

- $r_1 = r_2$: Let $a_k(r)$, $k \in \mathbb{N}$, be defined by the recurrence relation (5.9) (with a_0 and r given), and

$$\varphi(r,x) = x^r \sum_{k=0}^{\infty} a_k(r) x^k \,.$$

Then the computation leading to the recurrence relation (5.9) also yields that

$$\begin{aligned} &x^2\varphi_{xx}(r,x) + xp(x)\varphi_x(r,x) + q(x)\varphi(r,x) \\ &\quad = a_0 F(r)x^r + \sum_{k=1}^{\infty}\Big(F(k+r)a_k(r) + \Big(\sum_{j=0}^{k-1}\big[(j+r)p_{k-j} + q_{k-j}\big]a_j(r)\Big)x^{k+r} \\ &\quad = a_0(r-r_1)^2 x^r = 0\,, \end{aligned}$$

where φ_x and φ_{xx} denote the first and the second partial derivatives of φ w.r.t. x. Differentiating the equation above w.r.t. r variable at $r = r_1$, we find that

$$x^2\varphi_{xxr}(r,x) + xp(x)\varphi_{xr}(r,x) + q(x)\varphi_r(r,x) = \big[2a_0(r-r_1)^2x^r + a_0(r-r_1)^2x^r\log x\big]\Big|_{r=0} = 0\,.$$

If $\frac{\partial}{\partial r}\varphi_{xx} = \big(\frac{\partial \varphi}{\partial r}\big)_{xx}$ and $\frac{\partial}{\partial r}\varphi_x = \big(\frac{\partial \varphi}{\partial r}\big)_x$ (which in general is not true since it involves exchange of orders of limits), then the equation above implies that

$$x^2\Big(\frac{\partial \varphi}{\partial r}(r_1,\cdot)\Big)'' + xp(x)\Big(\frac{\partial \varphi}{\partial r}(r_1,\cdot)\Big)' + q(x)\varphi_r(r_1,\cdot) = 0\,.$$

In other words, assuming that $\frac{\partial}{\partial r}\varphi_{xx} = \big(\frac{\partial \varphi}{\partial r}\big)_{xx}$ and $\frac{\partial}{\partial r}\varphi_x = \big(\frac{\partial \varphi}{\partial r}\big)_x$, $y = \frac{\partial \varphi}{\partial r}(r_1,x)$ is also a solution to the ODE (5.5). Formally, we switch the order of the differentiation in r and the infinite sum to obtain that

$$\frac{\partial \varphi}{\partial r}(r_1,x) = x^{r_1}\log x\Big(\sum_{k=0}^{\infty} a_k(r)x^k\Big) + x^{r_1}\sum_{k=0}^{\infty} a_k'(r_1)x^k = y_1(x)\log x + \sum_{k=0}^{\infty} a_k'(r_1)x^{k+r_1}\,.$$

Now let us verify that

$$y_2(x) = y_1(x)\log x + \sum_{k=0}^{\infty} a_k'(r_1)x^{k+r_1} \tag{5.14}$$

is indeed a solution to (5.5) (if the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k'(r)x^k$ is not zero). We note that y_2 satisfies

$$\begin{aligned} xy_2' &= xy_1'(x)\log x + y_1(x) + \sum_{k=0}^{\infty}(k+r_1)a_k'(r)x^{k+r_1}\,, \\ x^2y_2'' &= x^2y_1''(x)\log x + 2xy'(x) - y_1(x) + \sum_{k=0}^{\infty}(k+r_1)(k+r_1-1)a_k'(r_1)x^{k+r_1}\,. \end{aligned}$$

Moreover, differentiating (5.8) w.r.t. r variable, we find that

$$\big[2(k+r_1)-1\big]a_k(r_1)+\sum_{j=0}^{k}p_{k-j}a_j(r_1)+\sum_{j=0}^{k}\big[p_{k-j}(j+r_1)+q_{k-j}\big]a_j'(r_1)=0 \qquad \forall\, k\in\mathbb{N}\cup\{0\}\,.$$

Therefore, by the fact that y_1 is a solution to (5.5), we have

$$\begin{aligned}
&x^2y_2''+xp(x)y_2'+q(x)y_2\\
&\quad=x^2y_1''(x)\log x+2xy_1'(x)-y_1(x)+\sum_{k=0}^{\infty}(k+r_1)(k+r-1)a_k'(r_1)x^{k+r_1}\\
&\qquad+xp(x)y_1'(x)\log x+p(x)y_1(x)+\Big(\sum_{k=0}^{\infty}p_kx^k\Big)\Big(\sum_{k=0}^{\infty}(k+r_1)a_k'(r_1)x^{k+r_1}\Big)\\
&\qquad+q(x)y_1(x)\log x+\Big(\sum_{k=0}^{\infty}q_kx^k\Big)\Big(\sum_{k=0}^{\infty}a_k'(r_1)x^{k+r_1}\Big)\\
&\quad=\sum_{k=0}^{\infty}\big[2(k+r_1)-1\big]a_k(r_1)x^{k+r_1}+\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}p_{k-j}a_j(r_1)\Big)x^{k+r_1}\\
&\qquad+\sum_{k=0}^{\infty}\Big((k+r_1)(k+r-1)a_k'(r_1)+\sum_{j=0}^{k}\big[p_{k-j}(j+r_1)+q_{k-j}\big]a_j'(r_1)\Big)x^{k+r_1}=0\,;
\end{aligned}$$

thus $y_2(x)$ is a solution to (5.5).

• $r_1-r_2=N\in\mathbb{N}$: using the recurrence relation (5.9) for $r=r_2$, by the fact that $F(r_2+N)=F(r_1)=0$ we cannot find $a_N(r_2)$ so that $a_{N+1}(r_2)$, $a_{N+2}(r_2)$ and so on cannot be determined.

1. Suppose that $\sum\limits_{j=0}^{N-1}(j+r)p_{N-j}+q_{N-j}$ is divisible by $r-r_2=r+N-r_1$. Since (5.9) implies that

$$(r-r_2)(r+N-r_2)a_N(r)=-\sum_{j=0}^{N-1}\big[(j+r)p_{N-j}+q_{N-j}\big]a_j(r)\,.$$

   we can compute $a_N(r_2)$ by

$$a_N(r_2)=\lim_{r\to r_2}a_N(r)=-\lim_{r\to r_2}\frac{\dfrac{\sum\limits_{j=0}^{N-1}(j+r)p_{N-j}+q_{N-j}}{r-r_2}}{\dfrac{F(r+N)}{r-r_2}}=-\frac{1}{N}\lim_{r\to r_2}\frac{\sum\limits_{j=0}^{N-1}(j+r)p_{N-j}+q_{N-j}}{r-r_2}\,;$$

   thus the recurrence relation can be used to determine $a_{N+1}(r_1)$, $a_{N+2}(r_1)$ and so on. In such a case, another solution can be written by (5.12) as well.

2. In general (which includes the case that $\sum\limits_{j=0}^{N-1}(j+r)p_{N-j}+q_{N-j}$ is divisible by $r-r_2$), we let

$$\varphi(r,x)=x^r\sum_{k=0}^{\infty}a_k(r)x^k\,,$$

   where $a_k(r)$ is given by the recurrence relation (5.9). Then

$$x^2\varphi_{xx}(r,x)+xp(x)\varphi_x(r,x)+q(x)\varphi(r,x)=a_0(r-r_1)(r-r_2)x^r\,.$$

Multiplying both sides of the equation above by $(r-r_2)$ then differentiating in r variable, with $\psi(r,x)$ denoting the function $(r-r_2)\varphi(r,x)$, we find that

$$x^2\psi_{xxr}(r_2,x) + xp(x)\psi_{xr}(r_2,x) + q(x)\psi_r(r_2,x) = 0\,,$$

which, as discussed before, under certain assumptions we find that

$$\psi_r(r_2,x) = \frac{\partial}{\partial r}\Big|_{r=r_2}\Big((r-r_2)\sum_{k=0}^{\infty} a_k(r)x^{k+r}\Big)$$

is also a solution to (5.5). Note that if $N \neq 1$, the recurrence relation (5.9) implies that

$$\lim_{r\to r_2}(r-r_2)a_1(r) = -\lim_{r\to r_2}\frac{\big(rp_1+q_1\big)(r-r_2)a_0}{F(r+1)} = 0\,.$$

Similarly, for $k < N$,

$$\lim_{r\to r_2}(r-r_2)a_k(r)\Big|_{r=r_2} = -\lim_{r\to r_2}\frac{\sum\limits_{j=0}^{k-1}\big[(j+r)p_{k-j}+q_{k-j}\big]a_j(r)(r-r_2)}{F(k+r)} = 0\,.$$

Now we consider $\lim\limits_{r\to r_2}(r-r_2)a_N(r)$. Since $F(r+N) = (r-r_2)(r+N-r_2)$, we have

$$(r-r_2)a_N(r) = -\frac{\sum\limits_{j=0}^{N-1}\big[(j+r)p_{k-j}+q_{k-j}\big]a_j(r)}{(r+N-r_2)}\,;$$

thus

$$\lim_{r\to r_2}(r-r_2)a_N(r) = -\frac{1}{N}\sum_{j=0}^{N-1}\big[(j+r_2)p_{k-j}+q_{k-j}\big]a_j(r_2)$$

which exists and might not vanish. Let $b_0 = \lim\limits_{r\to r_2}(r-r_2)a_N(r)$. Then for $k > N$, with b_k denoting the limit $\lim\limits_{r\to r_2}(r-r_2)a_{k+N}(r)$, we have

$$\begin{aligned} b_{k-N} &= \lim_{r\to r_2}(r-r_2)a_k(r) = -\lim_{r\to r_2}\frac{\sum\limits_{j=0}^{k-1}\big[(j+r)p_{k-j}+q_{k-j}\big]a_j(r)(r-r_2)}{F(k+r)} \\ &= -\lim_{r\to r_2}\frac{\sum\limits_{j=N}^{k-1}\big[(j+r)p_{k-j}+q_{k-j}\big]a_j(r)(r-r_2)}{F(k+r)} = -\frac{\sum\limits_{j=N}^{k-1}\big[(j+r_2)p_{k-j}+q_{k-j}\big]b_{j-N}}{F(k+r_2)} \\ &= -\frac{\sum\limits_{j=0}^{k-N-1}\big[(j+r_1)p_{k-j-N}+q_{k-j-N}\big]b_j}{F(k-N+r_1)} \end{aligned}$$

which implies that the sequence $\{b_j\}_{j=0}^{\infty}$ satisfies

$$F(k+r_1)b_k + \sum_{j=0}^{k-1}\big[(j+r_1)p_{k-j}+q_{k-j}\big]b_j = 0 \qquad \forall\, k\in\mathbb{N}\,.$$

As a consequence, by the fact that $\frac{a_k(r)}{a_0}$ is independent of a_0, we have $\frac{b_k}{b_0} = \frac{a_k(r_1)}{a_0}$ and

$$y_2(x) = \frac{\partial}{\partial r}\Big|_{r=r_2} \psi(r,x) = \frac{b_0}{a_0} y_1(x)\log x + \sum_{k=0}^{\infty} c_k(r_2) x^{k+r_2}\,, \tag{5.15}$$

where $b_0 = \lim_{r\to r_2}(r-r_2)a_N(r)$ and $c_k = \frac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_k(r)$.

Example 5.26. Find a series solution about 0 to $xy'' + y = 0$.

First, we note that

$$p_0 = \lim_{x\to 0} x\cdot\frac{0}{x} = 0 \qquad \text{and} \qquad q_0 = \lim_{x\to 0} x^2\cdot\frac{1}{x} = 0\,;$$

thus 0 is a regular singular point of the ODE and the indicial equation for 0 is $r(r-1) = 0$. There are two distinct roots $r_1 = 1$ and $r_2 = 0$ to the indicial equation for 0.

Let $\varphi(r,x) = \sum_{k=0}^{\infty} a_k(r)x^{k+r}$ be the solution to the ODE above. Then

$$\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k(r)x^{k+r-1} + \sum_{k=0}^{\infty} a_k(r)x^{k+r} = 0\,.$$

which implies that

$$a_0 r(r-1)x^{r-1} + \sum_{k=0}^{\infty}\big[(k+r+1)(k+r)a_{k+1}(r) + a_k(r)\big]x^{k+r} = 0\,.$$

Therefore,

$$a_{k+1}(r) = -\frac{1}{(k+r+1)(k+r)}a_k(r)\,; \tag{5.16}$$

thus

$$\begin{aligned} a_k(r) &= -\frac{1}{(k+r)(k+r-1)}a_{k-1}(r) = \frac{1}{(k+r)(k+r-1)^2(k+r-2)}a_{k-2}(r) \\ &= \cdots = \frac{(-1)^k}{(k+r)(k+r-1)^2\cdots(r+1)^2 r}a_0\,. \end{aligned}$$

Then $a_k(r_1) = \frac{(-1)^k}{(k+1)!k!}a_0$ which implies that a series solution is given by

$$y_1(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!k!}x^{k+1}\,.$$

We also note that the recurrence relation (5.16) can be obtained by (5.9): write the ODE as $x^2y'' + xy = 0$. Therefore, $p_k = 0$ for all $k \in \mathbb{N}\cup\{0\}$ and $q_k = \delta_{1k}$ for $k \in \mathbb{N}\cup\{0\}$, where $\delta_{\cdot\cdot}$ is the Kronecker delta. Using (5.9), we have

$$0 = F(k+r)a_k(r) + \sum_{j=0}^{k-1}\big[(j+r)p_{k-j} + q_{k-j}\big]a_j(r) = F(k+r)a_k(r) + a_{k-1}(r)\,.$$

We summarize the discussions above into the following

Theorem 5.27. *Let $x_0 = 0$ be a regular singular point of the differential equation* (5.5)*, and r_1 and r_2 be the roots of the indicial equation* (5.10) *with $r_1 \geqslant r_2$ if $r_1, r_2 \in \mathbb{R}$. Then there exists a series solution given by* (5.11)*.*

1. *If $r_1 - r_2 \notin \mathbb{N} \cup \{0\}$, then another solution is given by* (5.12)*.*

2. *If $r_1 = r_2$, then another solution is given by* (5.14)*.*

3. *If $r_1 - r_2 = N \in \mathbb{N}$, then another solution is given by* (5.15)*.*

In all three cases, the two solutions y_1 and y_2 form a fundamental set of solutions of the given differential equation.

5.6.2 The radius of convergence of series solutions

The radius of convergence of the series solution (5.7) cannot be guaranteed by Theorem 5.19; however, we have the following

Theorem 5.28 (Frobenius)**.** *If x_0 is a regular singular point of ODE* (5.1)*, then there exists at least one series solution of the form*

$$y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k \,,$$

where r is the largest root of the associated indicial equation. Moreover, the series solution converges for all $x \in 0 < x - x_0 < R$, where R is the distance from x_0 to the nearest other singular point (*real or complex*) *of* (5.1)*.*

5.7 Bessel's Equation

We consider three special cases of Bessel's equation

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \,, \tag{5.17}$$

where ν is a constant. It is easy to see that $x = 0$ is a regular singular point of (5.17) since

$$\lim_{x \to 0} x \cdot \frac{x}{x^2} = 1 = p_0 \qquad \text{and} \qquad \lim_{x \to 0} x^2 \cdot \frac{x^2 - \nu^2}{x^2} = -\nu^2 = q_0 \,.$$

Therefore, the indicial equation for the regular singular point $x = 0$ is

$$r(r-1) + r - \nu^2 = 0$$

which implies that $r = \pm\nu$. The ODE (5.17) is called ***Bessel's equation of order*** ν.

To find series solution to (5.17), we first note that in the case of Bessel's equation of order ν, $F(r) = r^2 - \nu^2$, $p(x) = 1$ (which implies that $p_0 = 1$ while $p_k = 0$ for all $k \in \mathbb{N}$) and $q(x) = x^2 - \nu^2$

(which implies that $q_0 = -\nu^2$ and $q_2 = 1$ and $q_k = 0$ otherwise). Therefore, the recurrence relation (5.9) implies that

$$\big[(k+r)^2 - \nu^2\big] a_k(r) + \sum_{j=0}^{k-1} q_{k-j} a_j(r) = 0 \qquad \forall\, k \in \mathbb{N}\,.$$

This implies that

$$\big[(1+r)^2 - \nu^2\big] a_1(r) = 0 \tag{5.18a}$$

$$\big[(k+r)^2 - \nu^2\big] a_k(r) + a_{k-2}(r) = 0 \qquad \forall\, k \geqslant 2 \tag{5.18b}$$

5.7.1 Bessel's Equation of Order Zero

Consider the case $\nu = 0$. Then the roots of the indicial equation are identical: $r_1 = r_2 = 0$. Using (5.18a), $a_1(r) \equiv 0$ (in a small neighborhood of 0) and (5.18b) implies that

$$a_k(r) = -\frac{1}{(k+r)^2} a_{k-2}(r) \qquad \forall\, k \geqslant 2\,; \tag{5.19}$$

thus $a_3(r) = a_5(r) = \cdots = a_{2m+1}(r) = \cdots = 0$ for all $m \in \mathbb{N}$. Note that $a_{2m-1}(r) = 0$ for all $m \in \mathbb{N}$ also implies that $a'_{2m-1}(r) = 0$ for all $m \in \mathbb{N}$.

On the other hand, recurrence relation (5.19) also implies that

$$\begin{aligned} a_{2m}(r) &= -\frac{1}{(2m+r)^2} a_{2m-2}(r) = \frac{1}{(2m+r)^2(2m+r-2)^2} a_{2m-4}(r) \\ &= \cdots = \frac{(-1)^{m-1}}{(2m+r)^2(2m+r-2)^2 \cdots (4+r)^2} a_2(r) \\ &= \frac{(-1)^m}{(2m+r)^2(2m+r-2)^2 \cdots (4+r)^2(2+r)^2} a_0\,; \end{aligned}$$

thus $a_{2m}(0) = \dfrac{(-1)^m}{2^{2m}(m!)^2} a_0$ and rearranging terms, we obtain that

$$\log \frac{(-1)^m a_{2m}(r)}{a_0} = -2\big[\log(2m+r) + \log(2m+r-2) + \cdots + \log(4+r) + \log(2+r)\big]\,.$$

Differentiating both sides above in r,

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2\Big[\frac{1}{2m+r} + \frac{1}{2m+r-2} + \cdots + \frac{1}{4+r} + \frac{1}{2+r}\Big]\,,$$

and evaluating the equation above at $r = 0$ we conclude that

$$a'_{2m}(0) = -H_m a_{2m}(0) = \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} a_0\,,$$

where $H_m = \sum\limits_{k=1}^{m} \dfrac{1}{k}$. As a consequence, the first series solution is given by

$$y_1(x) = \sum_{k=0}^{\infty} a_{2k}(0) x^{2k} = a_0\Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\Big]\,,$$

and the second solution is given by

$$y_2(x) = a_0\Big[J_0(x)\log x + \sum_{k=1}^{\infty}\frac{(-1)^k H_k x^{2k}}{2^{2k}(k!)^2}\Big],$$

where $J_0 = a_0^{-1}y_1$ is called the ***Bessel function of the first kind of order zero***. We note that y_1 and y_2 can be defined for all $x > 0$ since the radius of convergence of the series involved in y_1 and y_2 are infinite.

Any linear combinations of y_1 and y_2 is also a solution to Bessel's equation (5.17) of order zero. Consider the ***Bessel function of the second kind of order zero***

$$Y_0(x) = \frac{2}{\pi}\Big[\frac{1}{a_0}y_2(x) + (\gamma - \log 2)J_0(x)\Big], \tag{5.20}$$

where $\gamma = \lim\limits_{k\to\infty}(H_k - \log k) \approx 0.5772$ is called the ***Euler-Máscheroni constant***. Substituting for y_2 in (5.20), we obtain

$$Y_0(x) = \frac{2}{\pi}\Big[\Big(\gamma + \log\frac{x}{2}\Big)J_0(x) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_k}{2^{2k}(k!)^2}x^{2k}\Big], \qquad x > 0\,. \tag{5.21}$$

A general solution to Bessel's equation (5.17) of order zero then can be written as

$$y(x) = C_1 J_0(x) + C_2 Y_0(x)\,.$$

- Properties of J_0 and Y_0:

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}}\cos\Big(x - \frac{\pi}{4}\Big) \qquad \text{as } x \to \infty\,,$$
$$Y_0(x) \approx \sqrt{\frac{2}{\pi x}}\sin\Big(x - \frac{\pi}{4}\Big) \qquad \text{as } x \to \infty\,.$$

5.7.2 Bessel's Equation of Order One-Half

Now suppose that $\nu = \frac{1}{2}$ (thus $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$). To obtain solutions to Bessel's equation (5.17) of order one-half, we need to compute the coefficients $a_k(r)$ for all $k \in \mathbb{N}$ (given a_0), and $b_0 = \lim\limits_{r\to-\frac{1}{2}}(r - r_2)a_1(r)$ as well as $c_k = \frac{\partial}{\partial r}\Big|_{r=r_2}(r - r_2)a_k(r)$.

Using (5.18b), we find that

$$a_k(r) = \frac{-1}{(k+r)^2 - \frac{1}{4}}a_{k-2}(r) = \frac{-1}{(k+r+\frac{1}{2})(k+r-\frac{1}{2})}a_{k-2}(r) \qquad \forall\, k \geqslant 2\,,$$

while if $r \approx r_1 = \frac{1}{2}$, (5.18a) implies that $a_1(r) = 0$ which further implies that $a_3(r) = a_5(r) = \cdots = a_{2m-1}(r) = \cdots = 0$ for all $m \in \mathbb{N}$ if $r \approx \frac{1}{2}$. In particular, we have

$$a_{2m}(\tfrac{1}{2}) = \frac{(-1)^m a_0}{(2m+1)!} \quad \text{and} \quad a_{2m-1}(\tfrac{1}{2}) = 0 \qquad \forall\, m \in \mathbb{N}\,;$$

thus a series solution of (5.17) is

$$y_1(x) = a_0 x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} = a_0 x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = a_0 \frac{\sin x}{\sqrt{x}} .$$

The ***Bessel function of the first kind of order one-half*** is defined by (letting $a_0 = \sqrt{\frac{2}{\pi}}$ in the expression of y_1 above)

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{\sqrt{x}} = \sqrt{\frac{2}{\pi x}} \sin x .$$

Now we compute the limit of $(r - r_2)a_1(r)$ as r approaches r_2. Since (5.18a) implies that $(r + \frac{3}{2})(r + \frac{1}{2})a_1(r) = 0$, we have $(r - r_2)a_1(r) = 0$ for all $r \approx r_2 = -\frac{1}{2}$. Therefore,

$$b_0 = \lim_{r \to r_2} (r - r_2)a_1(r) = 0$$

which implies that there will be no logarithmic term in the second solution y_2 given by (5.12).

Now we compute $\frac{\partial}{\partial r}\Big|_{r=r_2} (r - r_2)a_k(r)$. Since

$$\begin{aligned} a_{2m}(r) &= \frac{-1}{(2m + r + \frac{1}{2})(2m + r - \frac{1}{2})} a_{2m-2}(r) = \cdots \\ &= \frac{(-1)^m}{(2m + r + \frac{1}{2})(2m + r - \frac{1}{2}) \cdots (2 + r + \frac{1}{2})(2 + r - \frac{1}{2})} a_0 \\ &= \frac{(-1)^m}{(2m + r + \frac{1}{2})(2m + r - \frac{1}{2}) \cdots (r + \frac{5}{2})(r + \frac{3}{2})} a_0 \end{aligned}$$

which implies that $|a'_{2m}(r_2)| < \infty$. Therefore,

$$c_{2m}(r_2) = \frac{\partial}{\partial r}\Big|_{r=r_2} (r - r_2)a_{2m}(r) = a_{2m}(r_2) = \frac{(-1)^m}{(2m)!} a_0 .$$

On the other hand, using (5.18a) again, we find that $a_1(r_2)$ is not necessary zero; thus we let a_1 be a free constant and use (5.18b) to obtain that

$$a_{2m+1}(r) = \frac{(-1)^m}{(2m + 1 + r + \frac{1}{2})(2m + 1 + r - \frac{1}{2}) \cdots (3 + r + \frac{1}{2})(3 + r - \frac{1}{2})} a_1 .$$

Since $|a'_{2m+1}(r_2)| < \infty$, we find that

$$c_{2m+1}(r_2) = \frac{\partial}{\partial r}\Big|_{r=r_2} (r - r_2)a_{2m+1}(r) = a_{2m}(r_2) = \frac{(-1)^m}{(2m+1)!} a_1 .$$

Therefore,

$$\begin{aligned} y_2(x) &= \sum_{k=0}^{\infty} c_k(r_2) x^{k+r_2} = x^{-\frac{1}{2}} \Big[a_0 \sum_{k=1}^{\infty} \frac{(-1)^m}{(2k)!} x^{2k} + a_1 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} \Big] \\ &= a_0 \frac{\cos x}{\sqrt{x}} + a_1 \frac{\sin x}{\sqrt{x}} . \end{aligned}$$

This produces the Bessel function of the second kind of order one-half

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x ,$$

and the general solution of Bessel's equation of order one-half can be written as $y = C_1 J_{\frac{1}{2}}(x) + C_2 J_{-\frac{1}{2}}(x)$.

5.7.3 Bessel's Equation of Order One

Now we consider the case that $\nu = 1$ (thus $r_1 = 1$ and $r_2 = -1$). Again, we need to compute $\{a_k(r_1)\}_{k=1}^{\infty}$, $\lim\limits_{r\to r_2}(r-r_2)a_2(r)$ and $c_k(r_2) = \dfrac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_k(r)$.

Note that (5.18a) implies that $a_1(r_1) = 0$ (which implies that $a_{2m-1}(r_1) = 0$ for all $m \in \mathbb{N}$). Moreover,

$$\begin{aligned}a_{2m}(r_1) &= \frac{-1}{(2m+2)2m}a_{2m-2}(r) = \frac{1}{(2m+2)(2m)^2(2m-2)}a_{2m-4}(r)\\ &= \cdots = \frac{(-1)^m}{(2m+2)(2m)^2(2m-4)^2\cdots 4^2\cdot 2}a_0 = \frac{(-1)^m}{2^{2m}(m+1)!m!}a_0\,;\end{aligned}$$

thus

$$y_1(x) = a_0 x\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k+1)!k!}x^{2k}\,.$$

Now we focus on finding b_0 and $\{c_k(r_2)\}_{k=0}^{\infty}$. Note that by (5.18a),

$$F(2+r)a_2(r) = -a_0\,;$$

thus $(r+1)a_2(r) = -\dfrac{1}{2(r+3)}$ which implies that $b_0 = \lim\limits_{r\to r_2}(r-r_2)a_2(r) = -\dfrac{a_0}{2}$.

To compute $\{c_k(r_2)\}_{k=0}^{\infty}$, we first note that (5.18a) implies that $a_1(r) \equiv 0$; thus we use (5.18b) to conclude that $a_{2m-1}(r) = 0$ for all $m \in \mathbb{N}$ and $r \approx r_2$. This implies that $c_{2m-1}(r_2) = 0$ for all $m \in \mathbb{N}$. On the other hand, for $m \in \mathbb{N}$ and $r \approx r_2$,

$$a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2\cdots(r+3)^2(r+1)}a_0\,;$$

thus

$$(r-r_2)a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2\cdots(r+3)^2}a_0\,.$$

Therefore, using the formula $\dfrac{d}{dr}f(r) = f(r)\dfrac{d}{dr}\log f(r)$ if $f(r) > 0$, we find that

$$\begin{aligned}c_{2m}(r_2) &= \frac{(-1)^{m+1}a_0}{(2m)(2m-2)^2\cdots 2^2}\Big[\frac{1}{2m+r+1}+\frac{2}{2m+r-1}+\cdots+\frac{2}{r+3}\Big]\Big|_{r=r_2}\\ &= \frac{(-1)^{m+1}a_0}{2^{2m-1}m!(m-1)!}\Big[\frac{1}{2m}+\frac{2}{2m-2}+\cdots+\frac{2}{2}\Big]\\ &= \frac{(-1)^{m+1}a_0}{2^{2m}m!(m-1)!}\Big[\frac{1}{m}+\frac{2}{m-1}+\cdots+\frac{2}{1}\Big] = \frac{(-1)^{m+1}\big(H_m+H_{m-1}\big)}{2^{2m}m!(m-1)!}a_0\,.\end{aligned}$$

Moreover, $c_0(r_2) = \dfrac{\partial}{\partial r}\Big|_{r=r_2}(r-r_2)a_0 = a_0$. Then the second solution to Bessel's equation of order one is

$$\begin{aligned}y_2(x) &= \frac{b_0}{a_0}y_1(x)\log x + \sum_{k=0}^{\infty}c_k(r_2)x^{k+r_2} = -J_1(x)\log x + x^{-1}\Big[a_0 + \sum_{k=1}^{\infty}c_{2k}(r_2)x^{2k}\Big]\\ &= -\frac{1}{2}y_1(x)\log x + \frac{a_0}{x}\Big[1-\sum_{k=1}^{\infty}\frac{(-1)^k(H_k+H_{k-1})}{2^{2k}k!(k-1)!}x^{2k}\Big]\,.\end{aligned}$$

This produces the ***Bessel function of the first kind of order one***:

$$J_1(x) \equiv \frac{1}{2}y_1(x) = \frac{x}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k+1)!k!}x^{2k}$$

and the ***Bessel function of the second kind of order one***:

$$Y_1(x) \equiv \frac{2}{\pi}\big[-y_2(x) + (\gamma - \log 2)J_1(x)\big]\,,$$

where γ is again the Euler-Máscheroni constant. The general solution to Bessel's equation of order one then can be written as

$$y = C_1J_1(x) + C_2Y_1(x)\,.$$

6 System of First Order Linear Equations

6.1 Introduction

There are several reasons that we should consider system of first order ODEs, and here we provide two of them.

1. In real life, a lot of phenomena can be modelled by system of first order ODE. For example, the LotkaVolterra equation or the predator-prey equation:

   $$\begin{aligned} p' &= \gamma p - \alpha pq\,, \\ q' &= \beta q + \delta pq\,. \end{aligned}$$

   in Example 1.10 can be used to described a predator-prey system. Let $\boldsymbol{x} \equiv (x_1, x_2) = (p, q)^{\mathrm{T}}$ and $\boldsymbol{F}(t, \boldsymbol{x}) = (\gamma x_1 - \alpha x_1x_2, \beta x_2 + \delta x_1x_2)^{\mathrm{T}}$. Then the Lotka-Volterra equation can also be written as

   $$\boldsymbol{x}'(t) = \boldsymbol{F}\big(t, \boldsymbol{x}(t)\big)\,. \tag{6.1}$$

2. Suppose that we are considering a scalar n-th order ODE

   $$y^{(n)}(t) = f\big(t, y(t), y'(t), \cdots, y^{(n-1)}(t)\big)\,.$$

   Let $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $\cdots$, $x_n(t) = y^{(n-1)}(t)$. Then $(x_1, \cdots, x_n)$ satisfies

   $$\begin{aligned} x_1'(t) &= x_2(t)\,, && (6.2a)\\ x_2'(t) &= x_3(t)\,, && (6.2b)\\ \vdots\ &= \ \vdots && (6.2c)\\ x_n'(t) &= f\big(t, x_1(t), x_2(t), \cdots, x_n(t)\big)\,. && (6.2d) \end{aligned}$$

   Let $\boldsymbol{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ be an n-vector, and $\boldsymbol{F}(t, \boldsymbol{x}) = \big(x_2, x_3, \cdots, x_n, f(t, x_1, x_2, \cdots, x_n)\big)^{\mathrm{T}}$ be a vector-valued function. Then (6.2) can also be written as (6.1).

Definition 6.1. The system of ODE (6.1) is said to be ***linear*** if $\boldsymbol{F}$ is of the form

$$\boldsymbol{F}(t,\boldsymbol{x}) = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$$

for some matrix-valued function $\boldsymbol{P} = \big[p_{ij}(t)\big]_{n\times n}$. (6.1) is said to be homogeneous if $\boldsymbol{g}(t) = 0$.

Example 6.2. Consider the second order ODE

$$y'' - y' - 2y = \sin t\,. \tag{6.3}$$

Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Then $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$ satisfies

$$\boldsymbol{x}'(t) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ \sin t \end{bmatrix}. \tag{6.4}$$

Therefore, the second order linear ODE (6.3) corresponds to a system of first order linear ODE (6.4). **Review**: to solve (6.3), we use the method of variation of parameters and assume that the solution to (6.3) can be written as

$$y(t) = u_1(t)e^{2t} + u_2(t)e^{-t}\,,$$

where $\{e^{2t}, e^{-t}\}$ is a fundamental set of (6.3). By the additional assumption $u_1'(t)e^{2t} + u_2'e^{-t} = 0$, we find that

$$\begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}.$$

Therefore, with $W(t)$ denoting the Wronskian of $\{e^{2t}, e^{-t}\}$, we have

$$u_1'(t) = \frac{1}{W(t)} \det\left(\begin{bmatrix} 0 & e^{-t} \\ \sin t & -e^{-t} \end{bmatrix}\right) = \frac{-e^{-t}\sin t}{-3e^t} = \frac{1}{3}e^{-2t}\sin t$$

and

$$u_2'(t) = \frac{1}{W(t)} \det\left(\begin{bmatrix} e^{2t} & 0 \\ 2e^{2t} & \sin t \end{bmatrix}\right) = \frac{e^{2t}\sin t}{-3e^t} = -\frac{1}{3}e^{t}\sin t$$

which further implies that a particular solution is

$$\begin{aligned} y(t) &= -\frac{2e^{-2t}\sin t + e^{-2t}\cos t}{15}e^{2t} + \frac{e^t\cos t - e^t \sin t}{6}e^{-t} \\ &= -\frac{2\sin t + \cos t}{15} + \frac{\cos t - \sin t}{6} = \frac{\cos t - 3\sin t}{10}\,. \end{aligned}$$

This particular solution provides a particular solution to (6.4):

$$\boldsymbol{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\cos t - 3\sin t}{10} \\ -\dfrac{\sin t + 3\cos t}{10} \end{bmatrix}.$$

Example 6.3. The ODE

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x} \tag{6.5}$$

is a system of first order linear homogeneous ODE. Suppose the initial condition is given by $\boldsymbol{x}(0) = (x_{10}, x_{20})^{\mathrm{T}}$.

1. Let $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$. Then

$$x_1'(t) = x_1(t) + x_2(t)\,, \tag{6.6a}$$

$$x_2'(t) = 4x_1(t) + x_2(t)\,. \tag{6.6b}$$

Note that (6.6a) implies $x_2 = x_1' - x_1$; thus replacing x_2 in (6.6) by $x_2 = x_1' - x_1$ we find that

$$x_1'' - x_1' = 4x_1 + x_1' - x_1 \qquad \text{or} \qquad x_1'' - 2x_1' - 3x_1 = 0\,.$$

Therefore, $x_1(t) = C_1 e^{3t} + C_2 e^{-t}$ and this further implies that $x_2(t) = 2C_1 e^{3t} - 2C_2 e^{-t}$; thus the solution to (6.5) can be expressed as

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}\,.$$

2. Let $\boldsymbol{x}_h(k) \approx \boldsymbol{x}(kh) = (x_1(kh), x_2(kh))^{\mathrm{T}}$ be the approximated value of $\boldsymbol{x}$ at the k-th step. Since

$$\boldsymbol{x}((k+1)h) \approx \boldsymbol{x}(kh) + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}_h(k)\,,$$

we consider the (explicit) Euler scheme

$$\boldsymbol{x}_h(k+1) = \boldsymbol{x}_h(k) + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}_h(k) = \left(\mathrm{Id} + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\right)^k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}\,,$$

and we expect that for $t > 0$ and $k = t/h$, then $\boldsymbol{x}_h(k) \to \boldsymbol{x}(t)$ as $h \to 0$.

To compute the k-th power of the matrix $\mathrm{Id} + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, we diagonize the matrix and obtain that

$$\mathrm{Id} + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1+h & h \\ 4h & 1+h \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1-h & 0 \\ 0 & 1+3h \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1};$$

thus

$$\left(\mathrm{Id} + h \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\right)^k = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} (1-h)^k & 0 \\ 0 & (1+3h)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1}.$$

As a consequence, using the limit $(1-h)^{\frac{t}{h}} \to e^{-t}$ and $(1+3h)^{\frac{t}{h}} \to e^{3t}$ as $t \to 0$, we find that

$$\begin{aligned} \boldsymbol{x}(t) = \lim_{h\to 0} \boldsymbol{x}_h\big(\tfrac{t}{h}\big) &= \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 2e^{-t} + 2e^{3t} & -e^{-t} + e^{3t} \\ -4e^{-t} + 4e^{3t} & 2e^{-t} + 2e^{3t} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 2x_{10} + x_{20} \\ 4x_{10} + 2x_{20} \end{bmatrix} e^{3t} + \frac{1}{4} \begin{bmatrix} 2x_{10} - x_{20} \\ -4x_{10} + 2x_{20} \end{bmatrix} e^{-t}\,. \end{aligned}$$

Choose $\boldsymbol{x}_0 = (1,2)^{\mathrm{T}}$ and $\boldsymbol{x}_0 = (1,-2)^{\mathrm{T}}$, we find that

$$\boldsymbol{x}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \boldsymbol{x}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

are both solution to (6.5).

Remark 6.4. For $a, b, c, d \in \mathbb{R}$ being given constants, suppose that x_1 and x_2 satisfy the system of first order linear ODE

$$x_1' = ax_1 + bx_2\,, \tag{6.7a}$$
$$x_2' = cx_1 + dx_2\,. \tag{6.7b}$$

Using (6.7a), we have $bx_2 = x_1' - ax_2$; thus (6.7b) implies that x_1 satisfies

$$x_1'' - (a+d)x_1' + (ad-bc)x_1 = 0\,.$$

We note that the characteristic equation for the ODE above is exactly the characteristic equation of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Moreover, suppose that $\lambda_1 \neq \lambda_2$ are distinct zeros of the characteristic equation, then

$$x_1(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\,.$$

Similarly, $x_2(t) = C_3 e^{\lambda_1 t} + C_4 e^{\lambda_2 t}$ for some C_3, C_4 satisfying

$$\lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} = (aC_1 + bC_3)e^{\lambda_1 t} + (aC_2 + bC_4)e^{\lambda_2 t}\,,$$
$$\lambda_1 C_3 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} = (cC_1 + dC_3)e^{\lambda_1 t} + (cC_2 + dC_4)e^{\lambda_2 t}\,.$$

Since $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ are linearly independent, we must have that C_1, C_2, C_3, C_4 satisfy

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = \lambda_2 \begin{bmatrix} C_2 \\ C_4 \end{bmatrix}\,.$$

In other words, $(C_1, C_3)^{\mathrm{T}}$ and $(C_2, C_4)^{\mathrm{T}}$ are the eigenvectors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ associated with eigenvalues λ_1 and λ_2, respectively. Therefore,

$$\boldsymbol{x}(t) = \begin{bmatrix} C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ C_3 e^{\lambda_1 t} + C_4 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} e^{\lambda_1 t} + \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} e^{\lambda_2 t} = \boldsymbol{u}_1 e^{\lambda_1 t} + \boldsymbol{u}_2 e^{\lambda_2 t}\,,$$

where $\boldsymbol{u}_1 = (C_1, C_3)^{\mathrm{T}}$ and $\boldsymbol{u}_2 = (C_2, C_4)^{\mathrm{T}}$.

6.2 Basic Theory of Systems of First Order Equations

Similar to Theorem 2.10, we have the following

Theorem 6.5. *Let $\boldsymbol{x}_0 = (x_{10}, x_{20}, \cdots, x_{n0})$ be a point in $\mathbb{R}^n$, $\mathcal{V} \subseteq \mathbb{R}^n$ be an open set containing $\boldsymbol{x}_0$, and $\boldsymbol{F} : (\alpha, \beta) \times \mathcal{V} \to \mathbb{R}^n$ be a vector-valued function of t and $\boldsymbol{x}$ such that $\boldsymbol{F} = (F_1, \cdots, F_n)$ and the partial derivative $\dfrac{\partial F_i}{\partial x_j}$ is continuous in $(\alpha, \beta) \times \mathcal{V}$ for all $i, j \in \{1, 2, \cdots, n\}$. Then in some interval $t \in (t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$, there exists a unique solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ to the initial value problem*

$$\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x}) \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0\,. \tag{6.8}$$

Moreover, if (6.8) is linear and $\mathcal{V} = \mathbb{R}^n$, then the solution exists throughout the interval (α, β).

The proof of this theorem is almost the same as the proof of Theorem 2.10 (by simply replacing $|\cdot|$ with $\|\cdot\|_{\mathbb{R}^n}$), and is omitted.

Corollary 6.6. *Let $(y_0, y_1, \cdots, y_{n-1})$ be a point in $\mathbb{R}^n$, $\mathcal{V} \subseteq \mathbb{R}^n$ be an open set containing $\boldsymbol{x}_0$, and $f : (\alpha, \beta) \times \mathcal{V} \to \mathbb{R}$ be a function such that $f : (\alpha, \beta) \times \mathcal{V} \to \mathbb{R}$ be real-valued function such that f and its partial derivatives $\dfrac{\partial f}{\partial y_i}$ is continuous in $(\alpha, \beta) \times \mathcal{V}$. Then in some interval $t \in (t_0 - h, t_0 + h) \subseteq (\alpha, \beta)$, there exists a unique solution $y = \varphi(t)$ to the initial value problem*

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)}) \qquad y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}\,.$$

In particular, the solution y is n-times continuously differentiable in $(t_0 - h, t_0 + h)$.

Theorem 6.7 (Principle of Superposition)**.** *If the vector $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are solutions of the linear system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, then the linear combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ is also a solution for any constants c_1 and c_2.*

Example 6.8. Consider the system of ODE

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x} \tag{6.5}$$

and note that $\boldsymbol{x}_1(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\boldsymbol{x}_2(t) = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ are solutions to this ODE; that is,

$$\boldsymbol{x}_1'(t) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} e^{3t} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}_1(t)$$

and

$$\boldsymbol{x}_2'(t) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}_2(t)\,.$$

Therefore, $y = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t)$ is also a solution to (6.5).

Theorem 6.9. *Let $\mathcal{M}_{n\times n}$ denote space of $n \times n$ real matrices, and $\boldsymbol{P} : (\alpha, \beta) \to \mathcal{M}_{n\times n}$ be a matrix-valued function. If the vector function $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ are linearly independent solutions to*

$$\boldsymbol{x}'(t) = \boldsymbol{P}(t)\boldsymbol{x}(t) \tag{6.9}$$

then each solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ to (6.9) can be expressed as a linear combination of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ in exact one way; that is, there exists a unique vector $(c_1, \cdots, c_n)$ such that

$$\boldsymbol{\varphi}(t) = c_1\boldsymbol{x}_1(t) + \cdots + c_n\boldsymbol{x}_n(t)\,. \tag{6.10}$$

Proof. By Theorem 6.5, for each $\boldsymbol{e}_i = (\underbrace{0, \cdots, 0}_{(i-1)\text{ slots}}, 1, 0, \cdots, 0)$, there exists a unique solution $\boldsymbol{x} = \boldsymbol{\varphi}_i(t)$ to (6.9) satisfying the initial data $\boldsymbol{x}(0) = \boldsymbol{e}_i$. The set $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n\}$ are linearly independent for otherwise there exists non-zero vectors $(c_1, \cdots, c_n)$ such that

$$c_1\boldsymbol{\varphi}_1(t) + c_2\boldsymbol{\varphi}_2(t) + \cdots + c_n\boldsymbol{\varphi}_n(t) = \boldsymbol{0}$$

which, by setting $t = 0$, would imply that $(c_1, c_2, \cdots, c_n) = \boldsymbol{0}$, a contradiction.

We note that $\{\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n\}$ is a fundamental set since every solution $\boldsymbol{x}(t)$ to (6.9) can be uniquely expressed by

$$\boldsymbol{x}(t) = x_1(0)\boldsymbol{\varphi}_1(t) + x_2(0)\boldsymbol{\varphi}_2(t) + \cdots + x_n(0)\boldsymbol{\varphi}_n(t)\,. \tag{6.11}$$

In fact, $\boldsymbol{x}(t)$ and $x_1(0)\boldsymbol{\varphi}_1(t) + \cdots + x_n(0)\boldsymbol{\varphi}_n(t)$ are both solutions to (6.9) satisfying the initial data

$$\boldsymbol{x}(0) = \big(x_1(0), \cdots, x_n(0)\big)^{\mathrm{T}};$$

thus by uniqueness of the solution, (6.11) holds.

Now, since $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ are solution to (6.9), we find that

$$\mathrm{span}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) \subseteq \mathrm{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\,.$$

Since $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ are linearly independent, $\dim\big(\mathrm{span}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)\big) = n$; thus by the fact that $\dim\big(\mathrm{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\big) = n$, we must have

$$\mathrm{span}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) = \mathrm{span}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)\,.$$

Therefore, every solution $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ of (6.9) can be (uniquely) expressed by (6.10). □

Definition 6.10. Let $\mathcal{P}(t) \in \mathcal{M}_{n\times n}$, and $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ be linearly independent solutions to (6.9). Then $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ is called a ***fundamental set*** of (6.9), the matrix $\boldsymbol{\Psi}(t) = \Big[[\boldsymbol{x}_1(t)] \vdots [\boldsymbol{x}_2(t)] \vdots \cdots \vdots [\boldsymbol{x}_n(t)]\Big]$ is called the ***fundamental matrix*** of (6.9), and $\boldsymbol{\varphi}(t) = c_1\boldsymbol{x}_1(t) + \cdots + c_n\boldsymbol{x}_n(t)$ is called the ***general solution*** of (6.9).

Theorem 6.11. *If $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n$ are solutions to* (6.9)*, then*

$$\det\Big(\Big[[\boldsymbol{\varphi}_1] \vdots [\boldsymbol{\varphi}_2] \vdots \cdots \vdots [\boldsymbol{\varphi}_n]\Big]\Big)$$

is either identically zero or else never vanishes.

Recall Theorem 4.3 that for a collection of solutions $\{\varphi_1, \cdots, \varphi_n\}$ to a n-th order ODE

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0\,,$$

the derivative of Wronskian $W(t) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$ satisfies

$$\frac{d}{dt}W(t) = -p_{n-1}(t)W(t)$$

which can be used to show that $W(t)$ is identically zero or else never vanishes. We use the same idea and try to find the derivative of the determinant $\mathrm{W}(t) \equiv \det\Big(\Big[[\boldsymbol{\varphi}_1] \vdots [\boldsymbol{\varphi}_2] \vdots \cdots \vdots [\boldsymbol{\varphi}_n]\Big]\Big)$. In view of Remark 6.4, we expect that we can derive

$$\frac{d}{dt}\mathrm{W}(t) = \mathrm{tr}(\boldsymbol{P})\mathrm{W}(t)\,.$$

Proof. Let $\mathrm{W}(t) \equiv \det\left(\Big[[\boldsymbol{\varphi}_1] \vdots [\boldsymbol{\varphi}_2] \vdots \cdots \vdots [\boldsymbol{\varphi}_n]\Big]\right)$, $\boldsymbol{P} = [p_{ij}]_{n\times n}$, and the i-th component of $\boldsymbol{\varphi}_j$ be $\varphi_j^{(i)}$; that is,

$$[\boldsymbol{\varphi}_j] = \left[\varphi_j^{(1)}, \cdots, \varphi_j^{(n)}\right]^{\mathrm{T}}.$$

Since $\varphi_j^{(i)\prime} = \sum_{k=1}^{n} p_{ik}\varphi_j^{(k)}$, using the properties of the determinants we find that

$$\begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(j-1)} & \varphi_2^{(j-1)} & \cdots & \cdots & \varphi_n^{(j-1)} \\ \varphi_1^{(j)\prime} & \varphi_2^{(j)\prime} & \cdots & \cdots & \varphi_n^{(j)\prime} \\ \varphi_1^{(j+1)} & \varphi_2^{(j+1)} & \cdots & \cdots & \varphi_n^{(j+1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \cdots & \varphi_n^{(n)} \end{vmatrix} = \begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(j-1)} & \varphi_2^{(j-1)} & \cdots & \cdots & \varphi_n^{(j-1)} \\ \sum\limits_{k=1}^{n} p_{jk}\varphi_1^{(k)} & \sum\limits_{k=1}^{n} p_{jk}\varphi_2^{(k)} & \cdots & \cdots & \sum\limits_{k=1}^{n} p_{jk}\varphi_n^{(k)} \\ \varphi_1^{(j+1)} & \varphi_2^{(j+1)} & \cdots & \cdots & \varphi_n^{(j+1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \cdots & \varphi_n^{(n)} \end{vmatrix}$$

$$\overset{\text{“row operations”}}{=} \begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(j-1)} & \varphi_2^{(j-1)} & \cdots & \cdots & \varphi_n^{(j-1)} \\ p_{jj}\varphi_1^{(j)} & p_{jj}\varphi_2^{(j)} & \cdots & \cdots & p_{jj}\varphi_n^{(j)} \\ \varphi_1^{(j+1)} & \varphi_2^{(j+1)} & \cdots & \cdots & \varphi_n^{(j+1)} \\ \vdots & \vdots & & & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \cdots & \varphi_n^{(n)} \end{vmatrix} = p_{jj}\mathrm{W}.$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\mathrm{W} &= \begin{vmatrix} \varphi_1^{(1)\prime} & \varphi_2^{(1)\prime} & \cdots & \varphi_n^{(1)\prime} \\ \varphi_1^{(2)} & \varphi_2^{(2)} & \cdots & \varphi_n^{(2)} \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix} + \begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \varphi_1^{(2)\prime} & \varphi_2^{(2)\prime} & \cdots & \varphi_n^{(2)\prime} \\ \varphi_1^{(3)} & \varphi_2^{(3)} & \cdots & \varphi_n^{(3)} \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \varphi_1^{(2)} & \varphi_2^{(2)} & \cdots & \varphi_n^{(2)} \\ \vdots & & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \\ \varphi_1^{(n)\prime} & \varphi_2^{(n)\prime} & \cdots & \varphi_n^{(n)\prime} \end{vmatrix} \\ &= (p_{11} + \cdots + p_{nn})\mathrm{W} = \mathrm{tr}(\boldsymbol{P})\mathrm{W}; \end{aligned}$$

thus

$$\mathrm{W}(t) = \exp\Big(\int_{t_0}^{t} \mathrm{tr}(\boldsymbol{P})(s)\,ds\Big)\mathrm{W}(t_0)$$

which implies that W is identically zero (if $\mathrm{W}(t_0)$ is zero) or else never vanishes (if $\mathrm{W}(t_0) \neq 0$). ▫

Definition 6.12. If $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n$ are n solutions to (6.9), the determinant

$$\mathrm{W}(\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n)(t) \equiv \det\left(\Big[[\boldsymbol{\varphi}_1] \vdots [\boldsymbol{\varphi}_2] \vdots \cdots \vdots [\boldsymbol{\varphi}_n]\Big]\right)$$

is called the ***Wronskian*** of $\{\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n\}$.

Theorem 6.13. *Let $\boldsymbol{u}, \boldsymbol{v} : (\alpha, \beta) \to \mathbb{R}^n$ be real-valued functions. If $\boldsymbol{x}(t) = \boldsymbol{u}(t) + i\boldsymbol{v}(t)$ is a solution to* (6.9)*, so are $\boldsymbol{u}$ and $\boldsymbol{v}$.*

Proof. Since $\boldsymbol{x}(t) = \boldsymbol{u}(t) + i\boldsymbol{v}(t)$ is a solution to (6.9), $\boldsymbol{x}'(t) - \boldsymbol{P}(t)\boldsymbol{x}(t) = \boldsymbol{0}$; thus

$$\begin{aligned}\boldsymbol{0} &= \boldsymbol{u}'(t) + i\boldsymbol{v}'(t) - \boldsymbol{P}(t)\big(\boldsymbol{u}(t) + i\boldsymbol{v}(t)\big) = \boldsymbol{u}'(t) + i\boldsymbol{v}'(t) - \boldsymbol{P}(t)\boldsymbol{u}(t) - i\boldsymbol{P}(t)\boldsymbol{v}(t) \\ &= \boldsymbol{u}'(t) - \boldsymbol{P}(t)\boldsymbol{u}(t) + i\big(\boldsymbol{v}'(t) - \boldsymbol{P}(t)\boldsymbol{v}(t)\big)\,.\end{aligned}$$

Since $\boldsymbol{u}'(t) - \boldsymbol{P}(t)\boldsymbol{u}(t)$ and $\boldsymbol{v}'(t) - \boldsymbol{P}(t)\boldsymbol{v}(t)$ are both real vectors, we must have

$$\boldsymbol{u}'(t) - \boldsymbol{P}(t)\boldsymbol{u}(t) = \boldsymbol{v}'(t) - \boldsymbol{P}(t)\boldsymbol{v}(t) = \boldsymbol{0}\,.$$

Therefore, $\boldsymbol{u}$ and $\boldsymbol{v}$ are both solutions to (6.9). ◻

6.3 Homogeneous Linear Systems with Constant Coefficients

In this section, we consider the equation

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)\,, \tag{6.12}$$

where $\boldsymbol{A}$ is a constant $n \times n$ matrix.

6.3.1 The case that $\boldsymbol{A}$ has distinct real eigenvalues

By Remark 6.4, it is natural to first look at the eigenvalues and eigenvectors of $\boldsymbol{A}$. Suppose that $\boldsymbol{A}$ has distinct real eigenvalues $\lambda_1, \cdots, \lambda_n$ with corresponding eigenvectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$. Let $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ and $\boldsymbol{P} = \Big[[\boldsymbol{v}_1] \vdots [\boldsymbol{v}_2] \vdots \cdots \vdots [\boldsymbol{v}_n]\Big]$. Then $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$ which implies that

$$\boldsymbol{x}'(t) = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}\boldsymbol{x}(t)\,.$$

Therefore, with $\boldsymbol{y}(t)$ denoting the vector $\boldsymbol{P}^{-1}\boldsymbol{x}(t)$, by the fact that $\boldsymbol{y}'(t) = \boldsymbol{P}\boldsymbol{x}'(t)$ (since $\boldsymbol{P}$ is a constant matrix), we have

$$\boldsymbol{y}'(t) = \boldsymbol{\Lambda}\boldsymbol{y}(t)\,. \tag{6.13}$$

In components, we obtain that for $1 \leqslant j \leqslant n$,

$$y_j'(t) = \lambda_j y_j(t)$$

if $\boldsymbol{y}(t) = \big(y_1(t), \cdots, y_n(t)\big)^{\mathrm{T}}$. As a consequence, if $\boldsymbol{y}(t_0) = \boldsymbol{y}_0 = (y_{01}, \cdots, y_{0n})^{\mathrm{T}}$ is given, we obtain that the solution to (6.13) (with initial data $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$) can be written as

$$\boldsymbol{y}(t) = \begin{bmatrix} e^{\lambda_1(t-t_0)}y_{01} \\ e^{\lambda_2(t-t_0)}y_{02} \\ \vdots \\ e^{\lambda_n(t-t_0)}y_{0n} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1(t-t_0)} & & & \\ & e^{\lambda_2(t-t_0)} & & \\ & & \ddots & \\ & & & e^{\lambda_n(t-t_0)} \end{bmatrix} \boldsymbol{y}_0\,;$$

thus the solution of (6.12) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ (which implies that $\boldsymbol{y}_0 = \boldsymbol{P}^{-1}\boldsymbol{x}_0$) can be written as

$$\boldsymbol{x}(t) = \boldsymbol{P}\boldsymbol{y}(t) = \boldsymbol{P}\begin{bmatrix} e^{\lambda_1(t-t_0)} & & & \\ & e^{\lambda_2(t-t_0)} & & \\ & & \ddots & \\ & & & e^{\lambda_n(t-t_0)} \end{bmatrix}\boldsymbol{P}^{-1}\boldsymbol{x}_0 . \tag{6.14}$$

Defining the exponential of an $n \times n$ matrix $\boldsymbol{M}$ by

$$e^{\boldsymbol{M}} = \mathbf{I}_{n\times n} + \boldsymbol{M} + \frac{1}{2!}\boldsymbol{M}^2 + \frac{1}{3!}\boldsymbol{M}^3 + \cdots + \frac{1}{k!}\boldsymbol{M}^k + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}\boldsymbol{M}^k ,$$

by the fact that $(t\boldsymbol{\Lambda})^k = \begin{bmatrix} (\lambda_1 t)^k & & \\ & \ddots & \\ & & (\lambda_n t)^k \end{bmatrix}$, we find that

$$e^{t\boldsymbol{\Lambda}} = \begin{bmatrix} \sum\limits_{k=0}^{\infty}\frac{1}{k!}(\lambda_1 t)^k & & \\ & \ddots & \\ & & \sum\limits_{k=0}^{\infty}\frac{1}{k!}(\lambda_n t)^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} .$$

Therefore, (6.14) implies that the solution to (6.12) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be expressed as

$$\boldsymbol{x}(t) = \boldsymbol{P}e^{(t-t_0)\boldsymbol{\Lambda}}\boldsymbol{P}^{-1}\boldsymbol{x}_0 .$$

Moreover, (6.14) also implies that the solution to (6.12) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be written as

$$\begin{aligned}
\boldsymbol{x}(t) &= \Big[[\boldsymbol{v}_1] \vdots \cdots \vdots [\boldsymbol{v}_n]\Big]\begin{bmatrix} e^{\lambda_1(t-t_0)} & & & \\ & e^{\lambda_2(t-t_0)} & & \\ & & \ddots & \\ & & & e^{\lambda_n(t-t_0)} \end{bmatrix}\begin{bmatrix} y_{01} \\ \vdots \\ y_{0n} \end{bmatrix} \\
&= \Big[e^{\lambda_1(t-t_0)}[\boldsymbol{v}_1] \vdots \cdots \vdots e^{\lambda_n(t-t_0)}[\boldsymbol{v}_n]\Big]\begin{bmatrix} y_{01} \\ \vdots \\ y_{0n} \end{bmatrix} \\
&= y_{01}e^{\lambda_1(t-t_0)}\boldsymbol{v}_1 + y_{02}e^{\lambda_2(t-t_0)}\boldsymbol{v}_2 + \cdots + y_{0n}e^{\lambda_n(t-t_0)}\boldsymbol{v}_n .
\end{aligned} \tag{6.15}$$

In other words, solutions to (6.12) are linear combination of vectors $\big\{e^{\lambda_1(t-t_0)}\boldsymbol{v}_1, \cdots, e^{\lambda_n(t-t_0)}\boldsymbol{v}_n\big\}$.

On the other hand, using that $t\boldsymbol{A} = \boldsymbol{P}(t\boldsymbol{\Lambda})\boldsymbol{P}^{-1}$, we have $(t\boldsymbol{A})^k = \boldsymbol{P}(t\boldsymbol{\Lambda})^k\boldsymbol{P}^{-1}$; thus the definition of exponential of matrices provides that

$$\begin{aligned}
e^{(t-t_0)\boldsymbol{A}} &= \sum_{k=0}^{\infty}\frac{1}{k!}\big((t-t_0)\boldsymbol{A}\big)^k = \sum_{k=0}^{\infty}\frac{1}{k!}\big(\boldsymbol{P}((t-t_0)\boldsymbol{\Lambda})^k\boldsymbol{P}^{-1}\big) = \boldsymbol{P}\Big[\sum_{k=0}^{\infty}\frac{1}{k!}\big((t-t_0)\boldsymbol{\Lambda}\big)^k\Big]\boldsymbol{P}^{-1} \\
&= \boldsymbol{P}e^{(t-t_0)\boldsymbol{\Lambda}}\boldsymbol{P}^{-1} .
\end{aligned}$$

Therefore, the solution to (6.12) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can also be expressed as

$$\boldsymbol{x}(t) = e^{(t-t_0)\boldsymbol{A}}\boldsymbol{x}_0 \,. \tag{6.16}$$

We remark that in contrast the solution to $x'(t) = ax(t)$, where a is a constant, can be written as

$$x(t) = e^{a(t-t_0)}x_0 \,,$$

where $x_0 = x(t_0)$ is the initial condition.

• **Stability**: Recall from Section 2.5 that an equilibrium solution to an autonomous ODE

$$y' = f(y) \tag{6.17}$$

is a time-independent solution $y(t) = y_e$ (thus y_e satisfies $f(y_e) = 0$), and the equilibrium solution $y(t) = y_e$ is said to be asymptotically stable if there exists $\delta > 0$ such that the solution y to (6.17) with initial data $y(t_0) = y_0$, where $|y_0 - y_e| < \delta$, satisfies

$$y(t) \to y_e \qquad \text{as} \quad t \to \infty \,,$$

while the equilibrium solution $y(t) = y_e$ is said to be unstable if there exists $r > 0$ such that for all $n \in \mathbb{N}$ there exists y_0 in the ball $B\big(y_e, \frac{1}{n}\big)$ such that the solution y to (6.17) with initial data $y(t_0) = y_0$ satisfies

$$\liminf_{t\to\infty} \big|y(t) - y_e\big| \geqslant r \,.$$

Similarly, we can look at the stability of an equilibrium solution to the autonomous system (6.12). An equilibrium solution to (6.12) is a time-independent solution $\boldsymbol{x}(t) = \boldsymbol{x}_e$ for some constant vector $\boldsymbol{x}_e$. In other words, $\boldsymbol{x}(t) = \boldsymbol{x}_e$ is an equilibrium solution if $\boldsymbol{A}\boldsymbol{x}_e = \boldsymbol{0}$. An equilibrium solution $\boldsymbol{x}(t) = \boldsymbol{x}_e$ to (6.12) is said to be asymptotically stable if there exists $\delta > 0$ such that the solution $\boldsymbol{x}$ to (6.12) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, where $|\boldsymbol{x}_0 - \boldsymbol{x}_e| < \delta$, satisfies

$$\boldsymbol{x}(t) \to \boldsymbol{x}_e \qquad \text{as} \quad t \to \infty \,.$$

If all the eigenvalues λ_j's are non-zero, then $y(t) = \boldsymbol{x}_e = \boldsymbol{0}$ is the only equilibrium solution, and using (6.15) we find that $\boldsymbol{x}_e = \boldsymbol{0}$ is an asymptotically stable equilibrium if and only if all the eigenvalues of $\boldsymbol{A}$ are negative.

• **Phase plane**: When $\boldsymbol{A} \in \mathcal{M}_{2\times 2}$, a special methodology, called the ***phase plane analysis***, can be applied to determine the stability of an equilibrium to (6.12). Note that for the case under consideration, (6.12) can, with $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$, be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

1. **Phase plane**: The x_1-x_2 plane is called the phase plane.

2. **Direction field**: The direction field of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is a normalized vector field $\boldsymbol{v}$ (that is, $|\boldsymbol{v}| = 1$) such that for each point $\boldsymbol{x}$ in the phase plane $\boldsymbol{v}(\boldsymbol{x})$ is in the same direction as the vector $\boldsymbol{A}\boldsymbol{x}$.

3. **Trajectory**: A trajectory of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is a solution curve $\boldsymbol{x}(t)$.

4. **Phase portrait**: A phase portrait of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is a collection of representative trajectories.

Example 6.14. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}$, and we consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. By looking at the direction field (on the next page), it is not difficult to see that $\mathbf{0}$ is not a stable equilibrium.

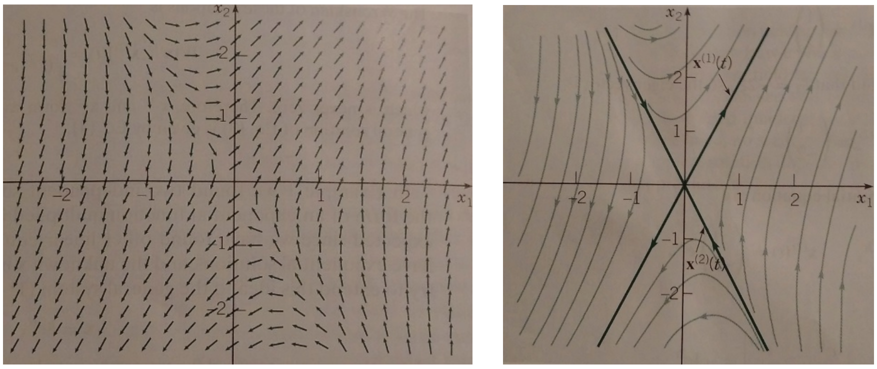


Figure 1: A direction field and a phase portrait of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$

On the other hand, we note that the eigenvalues of $\boldsymbol{A}$ are 3 and -1. Since not all the eigenvalues of $\boldsymbol{A}$ are negative, we also can conclude that $\mathbf{0}$ is not a stable equilibrium.

6.3.2 The case that $\boldsymbol{A}$ has complex eigenvalues

Now we consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ when $\boldsymbol{A}$ has complex eigenvalues.

Example 6.15. Find a fundamental set of real-valued solution of the system

$$\boldsymbol{x}' = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \boldsymbol{x} \,. \tag{6.18}$$

We first diagonalize the matrix $\boldsymbol{A} \equiv \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix}$ and find that

$$\begin{bmatrix} -1/2 & 1 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} -1/2+i & 0 \\ 0 & -1/2-i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} .$$

Therefore, Remark 6.4 implies that

$$\boldsymbol{x}_1(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1/2+i)t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}}(\cos t + i \sin t) = \begin{bmatrix} e^{-\frac{t}{2}} \cos t \\ -e^{-\frac{t}{2}} \sin t \end{bmatrix} + i \begin{bmatrix} e^{-\frac{t}{2}} \sin t \\ e^{-\frac{t}{2}} \cos t \end{bmatrix}$$

and

$$\boldsymbol{x}_2(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1/2-i)t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-\frac{t}{2}}(\cos t - i \sin t) = \begin{bmatrix} e^{-\frac{t}{2}} \cos t \\ -e^{-\frac{t}{2}} \sin t \end{bmatrix} - i \begin{bmatrix} e^{-\frac{t}{2}} \sin t \\ e^{-\frac{t}{2}} \cos t \end{bmatrix}$$

are both solutions to the ODE. By Theorem 6.13, $\boldsymbol{\varphi}_1(t) = \begin{bmatrix} e^{-\frac{t}{2}} \cos t \\ -e^{-\frac{t}{2}} \sin t \end{bmatrix}$ and $\boldsymbol{\varphi}_2(t) = \begin{bmatrix} e^{-\frac{t}{2}} \sin t \\ e^{-\frac{t}{2}} \cos t \end{bmatrix}$ are also solutions to (6.18).

To see the linear independence of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$, we note that the Wronskian of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ is

$$\mathrm{W}(t) = \begin{vmatrix} e^{-\frac{t}{2}}\cos t & e^{-\frac{t}{2}}\sin t \\ -e^{-\frac{t}{2}}\sin t & e^{-\frac{t}{2}}\cos t \end{vmatrix} = e^{-t}$$

which never vanishes. Therefore, $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\}$ is a fundamental set of (6.18).

Since $\det(\boldsymbol{A}) \neq 0$, $\mathbf{0}$ is the only equilibrium. By looking at the direction field and phase portrait of (6.18), we can image that $\mathbf{0}$ is a stable equilibrium.

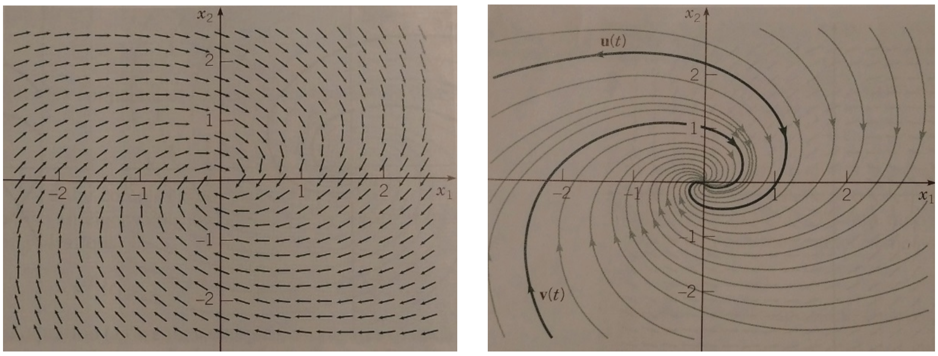


Figure 2: The direction field and phase portrait of (6.18)

In fact, since $\boldsymbol{\varphi}_1(t), \boldsymbol{\varphi}_2(t) \to 0$ as $t \to \infty$, any solution (which can be expressed as a linear combination of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$) to (6.18) converges to $\mathbf{0}$ as $t \to \infty$.

In general, if the constant matrix $\boldsymbol{A}$ has complex eigenvalues $r_\pm = \lambda \pm i\mu$ with corresponding eigenvectors $\boldsymbol{u}_\pm$. Then

$$(\boldsymbol{A} - r_\pm \mathbf{I})\boldsymbol{u}_\pm = \mathbf{0} \Leftrightarrow (\boldsymbol{A} - \overline{r_\pm}\mathbf{I})\overline{\boldsymbol{u}_\pm} = \mathbf{0} \Leftrightarrow (\boldsymbol{A} - r_\mp \mathbf{I})\overline{\boldsymbol{u}_\pm} = \mathbf{0}\,.$$

Therefore, $\boldsymbol{u}_-$ could be chosen as the complex conjugate of $\boldsymbol{u}_+$. Let $\boldsymbol{u}_+ = \boldsymbol{a} + i\boldsymbol{b}$ and $\boldsymbol{u}_- = \boldsymbol{a} - i\boldsymbol{b}$ be eigenvectors associated with r_+ and r_-, respective, where $\boldsymbol{a}, \boldsymbol{b}$ are real vectors. Let $\boldsymbol{x}_1(t) = \boldsymbol{u}_+ e^{r_+ t}$ and $\boldsymbol{x}_2(t) = \boldsymbol{a}_- e^{r_- t}$. Then $\boldsymbol{x}_1, \boldsymbol{x}_2$ are both solutions to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ since

$$\begin{aligned} \boldsymbol{x}_1'(t) &= r_+ \boldsymbol{u}_+ e^{r_+ t} = e^{r_+ t}(\boldsymbol{A}\boldsymbol{u}_+) = \boldsymbol{A}\boldsymbol{x}_1(t)\,, \\ \boldsymbol{x}_2'(t) &= r_- \boldsymbol{u}_- e^{r_- t} = e^{r_- t}(\boldsymbol{A}\boldsymbol{u}_-) = \boldsymbol{A}\boldsymbol{x}_2(t)\,. \end{aligned}$$

On the other hand, using the Euler identity we have

$$\begin{aligned} \boldsymbol{x}_1(t) &= (\boldsymbol{a} + i\boldsymbol{b})e^{(\lambda + i\mu)t} = (\boldsymbol{a} + i\boldsymbol{b})e^{\lambda t}(\cos\mu t + i\sin\mu t) \\ &= (\boldsymbol{a}\cos\mu t - \boldsymbol{b}\sin\mu t)e^{\lambda t} + i(\boldsymbol{a}\sin\mu t + \boldsymbol{b}\cos\mu t)e^{\lambda t}\,, \\ \boldsymbol{x}_2(t) &= (\boldsymbol{a} - i\boldsymbol{b})e^{(\lambda - i\mu)t} = (\boldsymbol{a} - i\boldsymbol{b})e^{\lambda t}(\cos\mu t - i\sin\mu t) \\ &= (\boldsymbol{a}\cos\mu t - \boldsymbol{b}\sin\mu t)e^{\lambda t} - i(\boldsymbol{a}\sin\mu t + \boldsymbol{b}\cos\mu t)e^{\lambda t}\,. \end{aligned}$$

Therefore, Theorem 6.13 implies that $\boldsymbol{\varphi}_1(t) \equiv (\boldsymbol{a}\cos\mu t - \boldsymbol{b}\sin\mu t)e^{\lambda t}$ and $\boldsymbol{\varphi}_2(t) \equiv (\boldsymbol{a}\sin\mu t + \boldsymbol{b}\cos\mu t)e^{\lambda t}$ are also solutions to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$.

Now suppose that $\boldsymbol{A}$ is an $n \times n$ matrix which has k distinct complex eigenvalues denoted by $r_\pm^{(1)}, r_\pm^{(2)}, \cdots, r_\pm^{(k)}$ and $n-2k$ distinct real eigenvalues $r_{2k+1}, \cdots, r_n$ with corresponding eigenvectors $\boldsymbol{u}_\pm^{(1)}$, $\boldsymbol{u}_\pm^{(2)}$, $\cdots$, $\boldsymbol{u}_\pm^{(k)}$, $\boldsymbol{u}_{2k+1}, \cdots, \boldsymbol{u}_k$, where

$$\boldsymbol{r}_\pm^{(j)} = \lambda_j \pm i\mu_j \text{ for some } \lambda_j, \mu_j \in \mathbb{R}, \text{ and } \boldsymbol{u}_+^{(j)} = \overline{\boldsymbol{u}_-^{(j)}} = \boldsymbol{a}^{(j)} + i\boldsymbol{b}^{(j)} .$$

Then the general solutions of $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is of the form

$$\boldsymbol{x}(t) = \sum_{j=1}^{k} \Big[C_1^{(j)}\big(\boldsymbol{a}^{(j)}\cos\mu_j t - \boldsymbol{b}^{(j)}\sin\mu_j t\big) + C_2^{(j)}\big(\boldsymbol{a}^{(j)}\sin\mu_j t + \boldsymbol{b}^{(j)}\cos\mu_j t\big)\Big]e^{\lambda_j t} + \sum_{j=2k+1}^{n} C_j \boldsymbol{u}_j e^{\lambda_j t} .$$

If $\boldsymbol{A}$ is a 2×2 matrix which has complex eigenvalues, then $\det(\boldsymbol{A}) \neq 0$; thus $\mathbf{0}$ is the only equilibrium of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Now we check the stability of this equilibrium. Let $\boldsymbol{u}, \boldsymbol{v}$ be given as above. Then the Wronskian of $\boldsymbol{u}, \boldsymbol{v}$ never vanishes. In fact,

$$\begin{aligned}
&W(\boldsymbol{u},\boldsymbol{v})(t) \\
&\quad = \begin{vmatrix} (a_1\cos\mu t - b_1\sin\mu t)e^{\lambda t} & (a_1\sin\mu t + b_1\cos\mu t)e^{\lambda t} \\ (a_2\cos\mu t - b_2\sin\mu t)e^{\lambda t} & (a_2\sin\mu t + b_2\cos\mu t)e^{\lambda t} \end{vmatrix} \\
&\quad = e^{2\lambda t}\big[(a_1\cos\mu t - b_1\sin\mu t)(a_2\sin\mu t + b_2\cos\mu t) - (a_2\cos\mu t - b_2\sin\mu t)(a_1\sin\mu t + b_1\cos\mu t)\big] \\
&\quad = e^{2\lambda t}(a_1 b_2 - a_2 b_1) \neq 0 ;
\end{aligned}$$

thus $\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly independent set. Moreover, Theorem 6.9 implies that every solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ can be expressed as a unique linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ (thus every solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ can be expressed as a unique linear combination of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$). Therefore, we immediately find that $\mathbf{0}$ is an asymptotically stable equilibrium if and only if $\lambda < 0$.

Example 6.16. Consider the two-mass three-spring system

$$\begin{aligned}
m_1\frac{d^2x_1}{dt^2} &= -(k_1+k_2)x_1 + k_2x_2 + F_1(t) , \\
m_2\frac{d^2x_2}{dt^2} &= k_2x_1 - (k_2+k_3)x_2 + F_2(t)
\end{aligned}$$

which is used to model the motion of two objects shown in the figure below.

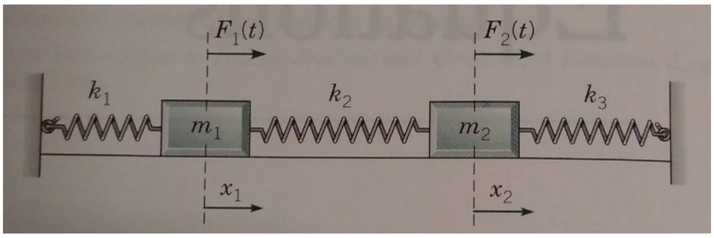


Figure 3: A two-mass three-spring system

Letting $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$ and $y_4 = x_2'$, we find that $\boldsymbol{y} = (y_1, y_2, y_3, y_4)^{\mathrm{T}}$ satisfies

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} & 0 & 0 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{F_1(t)}{m_1} \\ \dfrac{F_2(t)}{m_2} \end{bmatrix} .$$

Now suppose that $F_1(t) = F_2(t) = 0$, and $m_1 = 2$, $m_2 = \frac{9}{4}$, $k_1 = 1$, $k_2 = 3$, $k_3 = \frac{15}{4}$. Letting $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix}$, then $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$. The eigenvalue r of $\boldsymbol{A}$ satisfies

$$\det(\boldsymbol{A} - r\mathbf{I}) = \begin{vmatrix} -r & 0 & 1 & 0 \\ 0 & -r & 0 & 1 \\ -2 & \frac{3}{2} & -r & 0 \\ \frac{4}{3} & -3 & 0 & -r \end{vmatrix} = -r \begin{vmatrix} -r & 0 & 1 \\ \frac{3}{2} & -r & 0 \\ -3 & 0 & -r \end{vmatrix} + \begin{vmatrix} 0 & -r & 1 \\ -2 & \frac{3}{2} & 0 \\ \frac{4}{3} & -3 & -r \end{vmatrix}$$
$$= -r(-r^3 - 3r) + (6 - 2 + 2r^2) = r^4 + 5r^2 + 4 = 0\,.$$

Therefore, $\pm i, \pm 2i$ are eigenvalues of $\boldsymbol{A}$. Let $r_1 = i$, $r_2 = -i$, $r_3 = 2i$ and $r_4 = -2i$. Corresponding eigenvectors can be chosen as

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 6 \\ -8 \end{bmatrix}, \text{ and } \boldsymbol{u}_4 = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 0 \\ 6 \\ -8 \end{bmatrix}.$$

Therefore, with $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ denoting the vectors $(3, 2, 0, 0)^{\mathrm{T}}$, $(0, 0, 3, 2)^{\mathrm{T}}$, $(3, -4, 0, 0)^{\mathrm{T}}$ and $(0, 0, 6, -8)^{\mathrm{T}}$, respectively, the general solution to $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ is

$$\boldsymbol{y}(t) = C_1(\boldsymbol{a}\cos t - \boldsymbol{b}\sin t) + C_2(\boldsymbol{a}\sin t + \boldsymbol{b}\cos t) + C_3(\boldsymbol{c}\cos 2t - \boldsymbol{d}\sin 2t) + C_4(\boldsymbol{c}\sin 2t + \boldsymbol{d}\cos 2t)\,.$$

In particular,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} 3\cos t \\ 2\cos t \end{bmatrix} + C_2 \begin{bmatrix} 3\sin t \\ 2\sin t \end{bmatrix} + C_3 \begin{bmatrix} 3\cos 2t \\ -4\cos 2t \end{bmatrix} + C_4 \begin{bmatrix} 3\sin 2t \\ -4\sin 2t \end{bmatrix}.$$

- **Some conclusions**:

1. The phase space is four dimensional.
2. Each (C_1, C_2, C_3, C_4) corresponds to a trajectory in the phase space, and by the periodicity of the solution, each trajectory is a closed curve. Therefore, we know that the equilibrium $\mathbf{0}$ is not asymptotically stable.
3. When the the motion of the two masses corresponds to that $(C_3, C_4) = (0, 0)$, $x_2 = \frac{2}{3}x_1$; thus for these kind of motions the two masses move back and forth together and always moves in the same direction, but the second mass only move two-thirds as far as the first mass.
4. When the the motion of the two masses corresponds to that $(C_1, C_2) = (0, 0)$, $x_2 = -\frac{4}{3}x_1$; thus for these kind of motions the two masses move in opposite direction, and the second mass moves four-thirds as far as the first mass.
5. The two kinds of motions described above are called ***fundamental modes*** of vibration for the two-mass system, and for general initial conditions the solution is a combination of two fundamental modes.

6.3.3 The case that $\boldsymbol{A}$ has repeated eigenvalues

When $\boldsymbol{A}$ is diagonalizable, the discussion is pretty much the same as in the previous two sub-sections: if $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$, then the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be written as

$$\boldsymbol{x}(t) = \boldsymbol{P}\exp((t-t_0)\boldsymbol{\Lambda})\boldsymbol{P}^{-1}\boldsymbol{x}_0\,.$$

So we focus on the case that $\boldsymbol{A}$ is an $n\times n$ matrix which is not diagonalizable. In this case, there must be at least one eigenvalue λ of $\boldsymbol{A}$ such that the dimension of the eigenspace $\big\{\boldsymbol{v}\in\mathbb{C}^n \,\big|\, (\boldsymbol{A}-\lambda\mathbf{I})\boldsymbol{v} = \mathbf{0}\big\}$ is smaller than the algebraic multiplicity of λ.

Example 6.17. Let $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ and consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. We first compute the eigenvalues (and the corresponding eigenvectors) and find that 2 is the only eigenvalue (with algebraic multiplicity 2), while $\boldsymbol{u} = [1,-1]^{\mathrm{T}}$ is the only eigenvector associated with this eigenvalue. Therefore, $\boldsymbol{A}$ is not diagonalizable.

Let $\boldsymbol{x} = [x,y]^{\mathrm{T}}$. Then x, y satisfy

$$x' = x - y\,, \tag{6.19a}$$

$$y' = x + 3y\,. \tag{6.19b}$$

Using (6.19a) we obtain $y = x - x'$; thus applying this identity to (6.19b) we find that x satisfies

$$x' - x'' = x + 3(x - x') \qquad \text{or equivalently,} \qquad x'' - 4x' + 4x = 0\,.$$

The characteristic equation to the ODE above is $r^2 - 4r + 4 = 0$ (which should be the same as the characteristic equation for the matrix $\boldsymbol{A}$); thus 2 is the only zero. From the discussion in Section 3.4, we find that the solution to ODE (that x satisfies) is

$$x(t) = C_1 e^{2t} + C_2 t e^{2t}\,.$$

Using $y = x - x'$, we find that the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_1 e^{2t} + C_2 t e^{2t} \\ -(C_1 + C_2)e^{2t} - C_2 t e^{2t} \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t}\,.$$

Letting $\boldsymbol{v} = [0,1]^{\mathrm{T}}$, we have $\boldsymbol{x} = (C_1 + C_2 t)e^{2t}\boldsymbol{u} + C_2 e^{2t}\boldsymbol{v}$.

Given an large non-diagonalizable square matrix $\boldsymbol{A}$, it is almost impossible to carry out the same computation as in Example 6.17, so we need to find another systematic way to find the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. The following theorem states that $\boldsymbol{x}(t)$ given by (6.16) is always the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, even if $\boldsymbol{A}$ is not diagonalizable.

Theorem 6.18. *Let $\boldsymbol{A}$ be a square real constant matrix. Then the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is given by*

$$\boldsymbol{x}(t) = e^{(t-t_0)\boldsymbol{A}}\boldsymbol{x}_0\,. \tag{6.16}$$

Proof. Let $\boldsymbol{y}(t) = e^{(t-t_0)\boldsymbol{A}}\boldsymbol{x}_0$. Then

$$\begin{aligned}\boldsymbol{y}(t) &= \Big(\mathbf{I} + (t-t_0)\boldsymbol{A} + \frac{(t-t_0)^2}{2!}\boldsymbol{A}^2 + \cdots\Big)\boldsymbol{y}_0 \\ &= \boldsymbol{y}_0 + (t-t_0)\boldsymbol{A}\boldsymbol{y}_0 + \frac{(t-t_0)^2}{2!}\boldsymbol{A}^2\boldsymbol{y}_0 + \cdots + \frac{(t-t_0)^k}{k!}\boldsymbol{A}^k\boldsymbol{y}_0 + \cdots .\end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{y}'(t) &= \boldsymbol{A}\boldsymbol{y}_0 + (t-t_0)\boldsymbol{A}\boldsymbol{y}_0 + \cdots + \frac{(t-t_0)^{k-1}}{k!}\boldsymbol{A}^k\boldsymbol{y}_0 + \cdots \\ &= \boldsymbol{A}\Big(\mathbf{I} + (t-t_0)\boldsymbol{A} + \frac{(t-t_0)^2}{2!}\boldsymbol{A}^2 + \cdots\Big)\boldsymbol{y}_0 = \boldsymbol{A}\boldsymbol{y}\end{aligned}$$

which implies that $\boldsymbol{y}$ is a solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with initial data $\boldsymbol{y}(t_0) = e^{0\cdot\boldsymbol{A}}\boldsymbol{x}_0 = \boldsymbol{x}_0$. By the uniqueness of the solution, we know that the solution to (6.12) with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is given by (6.16). □

Having established Theorem 6.18, we now focus on how to compute the exponential of a square matrix if it is not diagonalizable.

For a 2×2 matrix $\boldsymbol{A}$ with repeated eigenvalue λ whose corresponding eigenvector is $\boldsymbol{u}$ (but not more linearly independent eigenvector), by Example 6.17 we can conjecture that the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = (C_1 + C_2 t)e^{\lambda t}\boldsymbol{u} + C_2 e^{\lambda t}\boldsymbol{v}$$

for some unknown vector $\boldsymbol{v}$. Now let us see what role $\boldsymbol{v}$ plays.

Since $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, we must have

$$\lambda(C_1 + C_2 t)e^{\lambda t}\boldsymbol{u} + C_2 e^{\lambda t}\boldsymbol{u} + C_2\lambda e^{\lambda t}\boldsymbol{v} = (C_1 + C_2 t)e^{\lambda t}\boldsymbol{A}\boldsymbol{u} + C_2 e^{\lambda t}\boldsymbol{A}\boldsymbol{v}\,.$$

By the fact that $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$ and C_2 is a general constant, the identity above implies that

$$\boldsymbol{u} = (\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{v}\,.$$

As a consequence, $\boldsymbol{v}$ satisfies $(\boldsymbol{A} - \lambda\mathbf{I})^2\boldsymbol{v} = 0$. Moreover, we must have $\boldsymbol{v} \nparallel \boldsymbol{u}$ (for otherwise $\boldsymbol{u} = \mathbf{0}$) which implies that $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent.

Let $\boldsymbol{P} = \big[\boldsymbol{u} \,\vdots\, \boldsymbol{v}\big]$, and $\boldsymbol{\Lambda} = \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$. Then $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{\Lambda}$. Since $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent, $\boldsymbol{P}$ is invertible; thus

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}\,.$$

Therefore, the same computations used in Section 6.3.1 shows that

$$e^{(t-t_0)\boldsymbol{A}} = \boldsymbol{P}e^{(t-t_0)\boldsymbol{\Lambda}}\boldsymbol{P}^{-1}\,.$$

Finally, taking $t_0 = 0$ (since the initial time could be translated to 0), then observing that

$$\Lambda^k = \begin{bmatrix}\lambda^k & k\lambda^{k-1}\\ 0 & \lambda^k\end{bmatrix}, \tag{6.20}$$

we conclude that

$$e^{t\boldsymbol{\Lambda}} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\boldsymbol{\Lambda}^k = \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{t^k}{k!}\lambda^k & \sum\limits_{k=1}^{\infty} \frac{t^k}{(k-1)!}\lambda^{k-1} \\ 0 & \sum\limits_{k=0}^{\infty} \frac{t^k}{k!}\lambda^k \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}. \tag{6.21}$$

Having obtained the identity above, using (6.16) one immediately see that the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is given by

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{u} \vdots \boldsymbol{v} \end{bmatrix} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

In the following, we develop a general theory to compute $e^{(t-t_0)\boldsymbol{A}}$ for a square matrix $\boldsymbol{A}$.

Definition 6.19. A square matrix $\boldsymbol{A}$ is said to be of ***Jordan canonical form*** if

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_2 & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{A}_\ell \end{bmatrix}, \tag{6.22}$$

where each $\boldsymbol{O}$ is zero matrix, and each $\boldsymbol{A}_i$ is a square matrix of the form $[\lambda]$ or

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

for some eigenvalue λ of $\boldsymbol{A}$.

We note that the diagonal elements of different $\boldsymbol{A}_i$ might be the same, and a diagonal matrix is of Jordan canonical form. Moreover, if $\boldsymbol{A}$ is of Jordan canonical form given by (6.22), then

$$\boldsymbol{A}^k = \begin{bmatrix} \boldsymbol{A}_1^k & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_2^k & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{A}_\ell^k \end{bmatrix} \quad \text{and} \quad e^{t\boldsymbol{A}} = \begin{bmatrix} e^{\boldsymbol{A}_1} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & e^{\boldsymbol{A}_2} & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & e^{\boldsymbol{A}_\ell} \end{bmatrix}. \tag{6.23}$$

Example 6.20. Let $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. Then $\boldsymbol{\Lambda}$ is of Jordan canonical form, and using (6.20) and (6.21) we conclude that

$$e^{t\boldsymbol{\Lambda}} = \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

Example 6.21. Let $\mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. Then $\mathbf{\Lambda}$ is of Jordan canonical form, and

$$\mathbf{\Lambda}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}.$$

Therefore,

$$e^{t\mathbf{\Lambda}} = \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{1}{k!} t^k \lambda^k & \sum\limits_{k=1}^{\infty} \frac{1}{(k-1)!} t^k \lambda^{k-1} & \sum\limits_{k=2}^{\infty} \frac{1}{2(k-2)!} t^k \lambda^{k-1} \\ 0 & \sum\limits_{k=0}^{\infty} \frac{1}{k!} t^k \lambda^k & \sum\limits_{k=1}^{\infty} \frac{1}{(k-1)!} t^k \lambda^{k-1} \\ 0 & 0 & \sum\limits_{k=0}^{\infty} \frac{1}{k!} t^k \lambda^k \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

In general, if $\mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda \end{bmatrix}$ is an $m \times m$ matrix, then with C^k_m denoting the number $\dfrac{k!}{m!(k-m)!}$ (if $k \geqslant m$, and 0 if $k < m$), we have

$$\mathbf{\Lambda}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & C^k_2 \lambda^{k-2} & \cdots & \cdots & C^k_{m-1}\lambda^{k-m+1} \\ 0 & \lambda^k & k\lambda^{k-1} & \ddots & \ddots & C^k_{m-2}\lambda^{k-m+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & 0 & \lambda^k & k\lambda^{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda^k \end{bmatrix}$$

(which can be shown by induction using Pascal's formula). As a consequence,

$$e^{t\mathbf{\Lambda}} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} & \cdots & \cdots & \frac{t^{m-1}}{(m-1)!} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \ddots & \ddots & \frac{t^{m-2}}{(m-2)!} e^{\lambda t} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & \cdots & \cdots & \cdots & 0 & e^{\lambda t} \end{bmatrix}. \tag{6.24}$$

The reason for introducing the Jordan canonical form and computing the exponential of matrices of Jordan canonical form is because of the following

Theorem 6.22. *Every square matrix is similar to a matrix of Jordan canonical form. In other words, if $\boldsymbol{A} \in \mathcal{M}_{n\times n}$, then there exists an invertible $n \times n$ matrix $\boldsymbol{P}$ and a matrix $\boldsymbol{\Lambda}$ of Jordan canonical form such that*

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}.$$

Given a Jordan decomposition $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$, we have $e^{t\boldsymbol{A}} = \boldsymbol{P}e^{t\boldsymbol{\Lambda}}\boldsymbol{P}^{-1}$ in which the exponential of $e^{t\boldsymbol{\Lambda}}$ can be obtained using (6.23) and (6.24); thus the computation of the exponential of a general square matrix $\boldsymbol{A}$ becomes easier as long as we know how to find the decomposition $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$.

- **How to obtain a Jordan decomposition of a square matrix $\boldsymbol{A}$?**

Definition 6.23 (Generalized Eigenvectors)**.** Let $\boldsymbol{A} \in \mathcal{M}_{n\times n}$. A vector $\boldsymbol{v} \in \mathbb{C}^n$ is called a generalized eigenvector of $\boldsymbol{A}$ associated with λ if $(\boldsymbol{A} - \lambda\mathbf{I})^p\boldsymbol{v} = \mathbf{0}$ for some positive integer p.

If $\boldsymbol{v}$ is a generalized eigenvector of $\boldsymbol{A}$ associated with λ, and p is the smallest positive integer for which $(\boldsymbol{A} - \lambda\mathbf{I})^p\boldsymbol{v} = \mathbf{0}$, then $(\boldsymbol{A} - \lambda\mathbf{I})^{p-1}\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ associated with λ. Therefore, λ is an eigenvalue of $\boldsymbol{A}$.

Definition 6.24 (Generalized Eigenspaces)**.** Let $\boldsymbol{A} \in \mathcal{M}_{n\times n}$ and λ be an eigenvalue of $\boldsymbol{A}$. The generalized eigenspace of $\boldsymbol{A}$ associated with λ, denoted by $\mathbf{K}_\lambda$, is the subset of $\mathbb{C}^n$ given by

$$\mathbf{K}_\lambda \equiv \left\{\boldsymbol{v} \in \mathbb{C}^n \,\middle|\, (\boldsymbol{A} - \lambda\mathbf{I})^p\boldsymbol{v} = \mathbf{0} \text{ for some positive integer } p\right\}.$$

- **The construction of Jordan decompositions**: Let $\boldsymbol{A} \in \mathcal{M}_{n\times n}$ be given.

Step 1: Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be all the eigenvalues of $\boldsymbol{A}$ with multiplicity $m_1, m_2, \cdots, m_k$. We first focus on how to determine the block

$$\boldsymbol{\Lambda}_j = \begin{bmatrix} \boldsymbol{\Lambda}_j^{(1)} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda}_j^{(2)} & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{\Lambda}_j^{(r_j)} \end{bmatrix},$$

whose diagonal is a fixed eigenvalue λ_j with multiplicity m_j for some $j \in \{1, 2, \cdots, k\}$, and the size of $\boldsymbol{\Lambda}_j^{(i)}$ is not smaller than the size of $\boldsymbol{\Lambda}_j^{(i+1)}$ for $i = 1, \cdots, r_j - 1$. Once all $\boldsymbol{\Lambda}_j's$ are obtained, then

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda}_2 & \ddots & \boldsymbol{O} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{\Lambda}_k \end{bmatrix}.$$

Step 2: Let $\mathbf{E}_j$ and $\mathbf{K}_j$ denote the eigenspace and the generalized eigenspace associated with λ_j, respectively. Then $r_j = \dim(\mathbf{E}_j)$ and $m_j = \dim(\mathbf{K}_j)$. Determine the smallest integer n_j such that

$$m_j = \dim\left(\mathrm{Ker}(\boldsymbol{A} - \lambda_j\mathbf{I})^{n_j}\right).$$

Find the value

$$p_j^{(\ell)} = \dim(\mathrm{Ker}(\boldsymbol{A} - \lambda_j \mathbf{I})^\ell) \quad \text{for} \quad \ell \in \{1, 2, \cdots, n_j\}$$

and set $p_j^{(0)} = 0$. Construct an $r_j \times n_j$ matrix whose entries only takes the value 0 or 1 and for each $\ell \in \{1, \cdots, n_j\}$ only the first $p_j^{(\ell)} - p_j^{(\ell-1)}$ components takes value 1 in the ℓ-th column of this matrix. Let $s_j^{(i)}$ be the sum of the i-th row of the matrix just obtained. Then $\boldsymbol{\Lambda}_j^{(i)}$ is a $s_j^{(i)} \times s_j^{(i)}$ matrix.

Step 3: Next, let us determine matrix $\boldsymbol{P}$. Suppose that

$$\boldsymbol{P} = \big[\boldsymbol{u}_1^{(1)} \vdots \cdots \vdots \boldsymbol{u}_1^{(m_1)} \vdots \boldsymbol{u}_2^{(1)} \vdots \cdots \vdots \boldsymbol{u}_2^{(m_2)} \vdots \boldsymbol{u}_3^{(1)} \vdots \cdots \vdots \boldsymbol{u}_k^{(n)}\big] .$$

Then $\boldsymbol{A}\big[\boldsymbol{u}_j^{(1)} \vdots \cdots \vdots \boldsymbol{u}_j^{(m_j)}\big] = \big[\boldsymbol{u}_j^{(1)} \vdots \cdots \vdots \boldsymbol{u}_j^{(m_j)}\big]\boldsymbol{\Lambda}_j$. Divide $\{\boldsymbol{u}_j^{(1)}, \cdots, \boldsymbol{u}_j^{(m_j)}\}$ into r_j groups:

$$\big\{\boldsymbol{u}_j^{(1)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)})}\big\}, \big\{\boldsymbol{u}_j^{(s_j^{(1)}+1)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)}+s_j^{(2)})}\big\}, \cdots, \text{ and } \big\{\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(r_j-1)}+1)}, \cdots, \boldsymbol{u}_j^{(m_j)}\big\} .$$

For each $\ell \in \{1, \cdots, r_j\}$, we let the ℓ-th group refer to the group of vectors

$$\Big\{\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell-1)}+1)}, \cdots, \boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell)})}\Big\} .$$

We then set up the first group by picking up an arbitrary non-zero vectors $\boldsymbol{v}_1 \in \mathrm{Ker}\big((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}}\big) \backslash \mathrm{Ker}\big((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}-1}\big)$ and let

$$\boldsymbol{u}_j^{(i)} = (\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(1)}-i} \boldsymbol{v}_1 \quad \text{for } i \in \{1, \cdots, s_j^{(1)} - 1\} .$$

Inductively, once the first ℓ groups of vectors are set up, pick up an arbitrary non-zero vectors $\boldsymbol{v}_{\ell+1} \in \mathrm{Ker}\big((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}}\big) \backslash \mathrm{Ker}\big((\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}-1}\big)$ such that $\boldsymbol{v}_{\ell+1}$ is not in the span of the vectors from the first ℓ groups, and define

$$\boldsymbol{u}_j^{(s_j^{(1)}+\cdots+s_j^{(\ell)}+i)} = (\boldsymbol{A} - \lambda_j \mathbf{I})^{s_j^{(\ell+1)}-i} \boldsymbol{v}_{\ell+1} \quad \text{for } i \in \{1, \cdots, s_j^{(\ell+1)} - 1\} .$$

This defines the $(\ell + 1)$-th group. Keep on doing so for all $\ell \leqslant r_j$ and for $j \in \{1, \cdots, k\}$, we complete the construction of $\boldsymbol{P}$.

Example 6.25. Find the Jordan decomposition of the matrix $\boldsymbol{A} = \begin{bmatrix} 4 & -2 & 0 & 2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & 0 & 6 \end{bmatrix}$.

If λ is an eigenvalue of $\boldsymbol{A}$, then λ satisfies

$$0 = \det(\boldsymbol{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & -2 & 0 & 2 \\ 0 & 6-\lambda & -2 & 0 \\ 0 & 2 & 2-\lambda & 0 \\ 0 & -2 & 0 & 6-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} 6-\lambda & -2 & 0 \\ 2 & 2-\lambda & 0 \\ -2 & 0 & 6-\lambda \end{vmatrix}$$

$$= (4-\lambda)\big[(6-\lambda)^2(2-\lambda) + 4(6-\lambda)\big] = (6-\lambda)(4-\lambda)\big[(6-\lambda)(2-\lambda) + 4\big]$$

$$= (\lambda - 4)^3(\lambda - 6) .$$

Let $\lambda_1 = 4$, $\lambda_2 = 6$, $m_1 = 3$ and $m_2 = 1$. Note that

$$\dim\big(\mathrm{Ker}(\boldsymbol{A} - 4\mathbf{I})\big) = 2 \quad \text{and} \quad \dim\big(\mathrm{Ker}(\boldsymbol{A} - 4\mathbf{I})^2\big) = 3\,.$$

Therefore, $n_1 = 2$ and $p_1^{(1)} = 2$, $p_1^{(2)} = 4$. We then construct the matrix according to Step 2 above, and the matrix is a 2×2 matrix given by $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. This matrix provides that $s_1 = 2$ and $s_2 = 1$; thus the block associated with the eigenvalue $\lambda = 4$, is $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore, $\boldsymbol{\Lambda} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$.

First, we note that the eigenvector associated with $\lambda = 6$ can be chosen as $(1,0,0,1)^{\mathrm{T}}$. Computing $\mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})\big)$ and $\mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})^2\big)$, we find that

$$\begin{aligned} \mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})\big) &= \mathrm{span}\big((1,0,0,0)^{\mathrm{T}}, (0,1,1,1)^{\mathrm{T}}\big)\,, \\ \mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})^2\big) &= \mathrm{span}\big((1,0,0,0)^{\mathrm{T}}, (0,1,0,2)^{\mathrm{T}}, (0,1,2,0)^{\mathrm{T}}\big)\,. \end{aligned}$$

We note that either $(0,1,0,2)^{\mathrm{T}}$ or $(0,1,2,0)^{\mathrm{T}}$ is in $\mathrm{Ker}\big((\boldsymbol{A} - 4\mathbf{I})\big)$, we can choose $\boldsymbol{v} = (0,1,0,2)^{\mathrm{T}}$. Then $(\boldsymbol{A} - 4\mathbf{I})\boldsymbol{v} = (2,2,2,2)^{\mathrm{T}}$. Finally, for the third column of $\boldsymbol{P}$ we can choose either $(1,0,0,0)^{\mathrm{T}}$ or $(0,1,1,1)^{\mathrm{T}}$ (or even their linear combination) since these vectors are not in the span of $(2,2,2,2)^{\mathrm{T}}$ and $(0,1,0,2)$. Therefore,

$$\boldsymbol{P} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{P} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

satisfies $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$.

Example 6.26. Let $\boldsymbol{A}$ be given in Example 6.25, and consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Let $\boldsymbol{u}_1 = (2,2,2,2)^{\mathrm{T}}$, $\boldsymbol{u}_2 = (0,1,0,2)^{\mathrm{T}}$, $\boldsymbol{u}_3 = (1,0,0,0)^{\mathrm{T}}$ and $\boldsymbol{u}_4 = (1,0,0,1)^{\mathrm{T}}$. Then the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is given by

$$\begin{aligned} \boldsymbol{x}(t) &= \big[\boldsymbol{u}_1 \vdots \boldsymbol{u}_2 \vdots \boldsymbol{u}_3 \vdots \boldsymbol{u}_4\big] e^{t\boldsymbol{\Lambda}} (\boldsymbol{P}^{-1}\boldsymbol{x}_0) \\ &= \big[\boldsymbol{u}_1 \vdots \boldsymbol{u}_2 \vdots \boldsymbol{u}_3 \vdots \boldsymbol{u}_4\big] \begin{bmatrix} e^{4t} & te^{4t} & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & e^{6t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \\ &= \big[\boldsymbol{u}_1 \vdots \boldsymbol{u}_2 \vdots \boldsymbol{u}_3 \vdots \boldsymbol{u}_4\big] \begin{bmatrix} C_1 e^{4t} + C_2 te^{4t} \\ C_2 e^{4t} \\ C_3 e^{4t} \\ C_4 e^{6t} \end{bmatrix} \\ &= (C_1 e^{4t} + C_2 te^{4t})\boldsymbol{u}_1 + C_2 e^{4t}\boldsymbol{u}_2 + C_3 e^{4t}\boldsymbol{u}_3 + C_4 e^{6t}\boldsymbol{u}_4\,, \end{aligned}$$

where $\boldsymbol{\Lambda}$ is given in Example 6.25, $\boldsymbol{x}_0$ is the value of $\boldsymbol{x}$ at $t = 0$ (which can be arbitrarily given), and $(C_1, C_2, C_3, C_4)^{\mathrm{T}} = \boldsymbol{P}^{-1}\boldsymbol{x}_0$.

Example 6.27. Let $\boldsymbol{A} = \begin{bmatrix} a & 0 & 1 & 0 & 0 \\ 0 & a & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 1 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}$. Then the characteristic equation of $\boldsymbol{A}$ is $(a-\lambda)^5$; thus $\lambda = a$ is the only eigenvalue of $\boldsymbol{A}$. First we compute the kernel of $(\boldsymbol{A} - a\mathbf{I})^p$ for various p. With $\boldsymbol{e}_i = (\underbrace{0, \cdots, 0}_{(i-1)\text{-slots}}, 1, 0, \cdots, 0)^{\mathrm{T}}$ denoting the i-th vector in the standard basis of $\mathbb{R}^5$, we find that

$$\begin{aligned} \operatorname{Ker}((\boldsymbol{A} - a\mathbf{I})) &= \{\boldsymbol{e}_1 \,|\, x_1, x_2 \in \mathbb{R}\} = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2)\,, \\ \operatorname{Ker}((\boldsymbol{A} - a\mathbf{I})^2) &= \{(x_1, x_2, x_3, x_4, 0)^{\mathrm{T}} \,|\, x_1, x_2, x_3, x_4 \in \mathbb{R}\} = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)\,, \\ \operatorname{Ker}((\boldsymbol{A} - a\mathbf{I})^3) &= \mathbb{R}^5 = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5)\,. \end{aligned}$$

The matrix obtained by Step 2 is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which implies that the two Jordan blocks is of size 3×3 and 2×2. Therefore,

$$\Lambda = \begin{bmatrix} a & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

We note that $\boldsymbol{e}_5 \in \operatorname{Ker}\big((\boldsymbol{A} - a\mathbf{I})^3\big) \backslash \operatorname{Ker}\big((\boldsymbol{A} - a\mathbf{I})^2\big)$; thus the first three column of $\boldsymbol{P}$ can be chosen as

$$\boldsymbol{P}(1:3) = \big[(\boldsymbol{A} - a\mathbf{I})^2 \boldsymbol{e}_5 \,\vdots\, (\boldsymbol{A} - a\mathbf{I})\boldsymbol{e}_5 \,\vdots\, \boldsymbol{e}_5\big] = \big[\boldsymbol{e}_1 \,\vdots\, \boldsymbol{e}_3 \,\vdots\, \boldsymbol{e}_5\big]\,.$$

To find the last two columns, we try to find a vector $\boldsymbol{w} \in \operatorname{Ker}\big((\boldsymbol{A} - a\mathbf{I})^2\big) \backslash \operatorname{Ker}\big((\boldsymbol{A} - a\mathbf{I})\big)$ so that $\boldsymbol{w}$ is not in the span of $\{\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_5\}$. Therefore, we may choose $\boldsymbol{w} = \boldsymbol{e}_4$; thus the last two columns of $\boldsymbol{P}$ is

$$\boldsymbol{P}(4:5) = \big[(\boldsymbol{A} - a\mathbf{I}\boldsymbol{e}_4 \,\vdots\, \boldsymbol{e}_4\big] = \big[\boldsymbol{e}_2 \,\vdots\, \boldsymbol{e}_4\big]$$

which implies that

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Example 6.28. Let $\boldsymbol{A}$ be given in Example 6.25, and consider the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Following the procedure in Example 6.26, we find that the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is given by

$$\begin{aligned} \boldsymbol{x}(t) &= \big[\boldsymbol{e}_1 \,\vdots\, \boldsymbol{e}_3 \,\vdots\, \boldsymbol{e}_5 \,\vdots\, \boldsymbol{e}_2 \,\vdots\, \boldsymbol{e}_4\big] \begin{bmatrix} e^{at} & te^{at} & \frac{t^2}{2}e^{at} & 0 & 0 \\ 0 & e^{at} & te^{at} & 0 & 0 \\ 0 & 0 & e^{at} & 0 & 0 \\ 0 & 0 & 0 & e^{at} & te^{at} \\ 0 & 0 & 0 & 0 & e^{at} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \\ &= \big(C_1 e^{at} + C_2 te^{at} + \frac{C_3}{2} t^2 e^{at}\big)\boldsymbol{e}_1 + (C_2 e^{at} + C_3 te^{at})\boldsymbol{e}_3 + C_3 e^{at}\boldsymbol{e}_5 + (C_4 e^{at} + C_5 te^{at})\boldsymbol{e}_2 + C_5 e^{at}\boldsymbol{e}_4\,. \end{aligned}$$

6.4 Fundamental Matrices

In Definition 6.10 we have talked about the fundamental matrix of system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$. It is defined as a square matrix whose columns form an linearly independent set of solutions to the ODE $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$. Let $\boldsymbol{\Psi}$ be a fundamental matrix of $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$. Since each column of $\boldsymbol{\Psi}$ is a solution to the ODE, we must have

$$\boldsymbol{\Psi}'(t) = \boldsymbol{P}(t)\boldsymbol{\Psi}(t)\,.$$

By the linearly independence of columns of $\boldsymbol{\Psi}$, we must have

$$\boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t)^{-1} = \boldsymbol{P}(t) \qquad \text{for all } t \text{ in the interval of interest.} \tag{6.25}$$

A special kind of fundamental matrix $\boldsymbol{\Phi}$, whose initial value $\boldsymbol{\Phi}(t_0)$ is the identity matrix, is in particular helpful for constructing solutions to

$$\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}\,, \tag{6.26a}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0\,. \tag{6.26b}$$

In fact, if $\boldsymbol{\Phi}$ is a fundamental matrix of system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$ satisfying $\boldsymbol{\Phi}(t_0) = \mathbf{I}$, then the solution to (6.26) is given by

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}_0\,.$$

It should be clear to the readers that the i-th column of $\boldsymbol{\Phi}$ is the solution to

$$\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}\,,$$

$$\boldsymbol{x}(t_0) = \boldsymbol{e}_i\,,$$

where $\boldsymbol{e}_i = (\underbrace{0,\cdots,0}_{(i-1)\text{-slots}},1,0,\cdots,0)^{\mathrm{T}}$ is the i-th vector in the standard basis of $\mathbb{R}^n$ (here we assume that the size of $\boldsymbol{P}$ is $n \times n$). Moreover, for each fundamental matrix $\boldsymbol{\Psi}$ of (6.26a), we have the relation

$$\boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Psi}(t_0)\,.$$

Therefore, given a fundamental matrix $\boldsymbol{\Psi}$, we can easily construct the fundamental matrix $\boldsymbol{\Phi}(t)$ by

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\,.$$

Caution: Based on the discussions above and the information that the solution to the scalar equation $x' = p(t)x$ with initial data $x(t_0) = x_0$ is $x(t) = \exp\Big(\int_{t_0}^t p(s)\,ds\Big)x_0$, one might start guessing that the solution to (6.26) is

$$\boldsymbol{x}(t) = \exp\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)\boldsymbol{x}_0\,. \tag{6.27}$$

This is in fact **NOT TRUE** because in general $\boldsymbol{P}(s)\boldsymbol{P}(t) \neq \boldsymbol{P}(t)\boldsymbol{P}(s)$. Nevertheless, if $\boldsymbol{P}(s)\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{P}(s)$ for all s and t, then the solution to (6.26) is indeed given by (6.27). To see this, we first notice that

$$\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big) = \int_{t_0}^t \boldsymbol{P}(t)\boldsymbol{P}(s)\,ds = \int_{t_0}^t \boldsymbol{P}(s)\boldsymbol{P}(t)\,ds = \Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)\boldsymbol{P}(t)\,;$$

thus

$$\begin{aligned}\frac{d}{dt}\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^k &= \boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-1} + \Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-2} + \cdots \\ &\quad + \Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-2}\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big) + \Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-1}\boldsymbol{P}(t) \\ &= k\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-1}.\end{aligned}$$

Therefore, the function given by (6.27) satisfies that

$$\begin{aligned}\frac{d}{dt}\exp\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)\boldsymbol{x}_0 &= \frac{d}{dt}\Big[\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^k\Big]\boldsymbol{x}_0 = \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\boldsymbol{P}(t)\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^{k-1}\boldsymbol{x}_0 \\ &= \boldsymbol{P}(t)\Big(\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big)^k\Big)\boldsymbol{x}_0 = \boldsymbol{P}(t)\exp\Big(\int_{t_0}^t \boldsymbol{P}(s)\,ds\Big).\end{aligned}$$

On the other hand, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. As a consequence, $\boldsymbol{x}(t)$ given by (6.27) is the solution to (6.26).

Now suppose that $\boldsymbol{P}(t) = \boldsymbol{A}$ is time-independent. Then by Theorem 6.18 we find that the fundamental matrix $\boldsymbol{\Phi}(t)$ is given by

$$\boldsymbol{\Phi}(t) = \boldsymbol{P}e^{(t-t_0)\boldsymbol{\Lambda}}\boldsymbol{P}^{-1},$$

where $\boldsymbol{P\Lambda P}^{-1}$ is a Jordan decomposition of $\boldsymbol{A}$. Moreover,

$$\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s) \qquad \forall\, t, s \in \mathbb{R}. \tag{6.28}$$

To see this, let t_1, t_2 be given real number, and $\boldsymbol{x}_0 \in \mathbb{R}^n$ be a vector. By the existence and uniqueness theorem (Theorem 6.5), the solution to system $\boldsymbol{x}' = \boldsymbol{Ax}$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is given by $\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}_0$ for all $t \in \mathbb{R}$.

On the other hand, again by the uniqueness of the solution, the solution $\boldsymbol{\varphi}_1$ to

$$\begin{aligned}\boldsymbol{\varphi}' &= \boldsymbol{A\varphi}, \\ \boldsymbol{\varphi}(t_0) &= \boldsymbol{x}(t_1),\end{aligned}$$

and the solution $\boldsymbol{\varphi}_2$ to

$$\begin{aligned}\boldsymbol{\varphi}' &= \boldsymbol{A\varphi}, \\ \boldsymbol{\varphi}(t_0) &= \boldsymbol{x}(t_2),\end{aligned}$$

satisfy that $\boldsymbol{\varphi}_1(t) = \boldsymbol{x}(t-t_0+t_1)$ and $\boldsymbol{\varphi}_2(t) = \boldsymbol{x}(t-t_0+t_2)$. Moreover, using the fundamental matrix $\boldsymbol{\Phi}$ we also have $\boldsymbol{\varphi}_1(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(t_1)$ and $\boldsymbol{\varphi}_2(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(t_2)$. Therefore,

$$\boldsymbol{\Phi}(t_2)\boldsymbol{\Phi}(t_1)\boldsymbol{x}_0 = \boldsymbol{\Phi}(t_2)\boldsymbol{x}(t_1) = \boldsymbol{\varphi}_1(t_2) = \boldsymbol{x}(t_1+t_2-t_0) = \boldsymbol{\varphi}_2(t_1) = \boldsymbol{\Phi}(t_1)\boldsymbol{\Phi}(t_2)\boldsymbol{x}_0.$$

Since $\boldsymbol{x}_0$ is arbitrary, we must have $\boldsymbol{\Phi}(t_2)\boldsymbol{\Phi}(t_1) = \boldsymbol{\Phi}(t_1)\boldsymbol{\Phi}(t_2)$; thus (6.28) is concluded.

6.5 Non-homogeneous Linear Systems

Now we consider the non-homogeneous linear system

$$\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)\,, \tag{6.29a}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0\,, \tag{6.29b}$$

for some non-zero vector-valued forcing $\boldsymbol{g}$. As in Definition 3.13 we said that a vector-valued function $\boldsymbol{x}_p(t)$ is called a ***particular solution*** to (6.29a) if $\boldsymbol{x}_p$ satisfies (6.29a). As long as a particular solution to (6.29a) is obtained, then the general solution to (6.29a) is given by

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{C} + \boldsymbol{x}_p(t)\,,$$

where $\boldsymbol{\Psi}$ is a fundamental matrix of $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, and $\boldsymbol{C}$ is an arbitrary constant vector. to satisfy the initial data (6.29b), we let $\boldsymbol{C} = \boldsymbol{\Psi}(t_0)^{-1}\big(\boldsymbol{x}_0 - \boldsymbol{x}_p(t_0)\big)$ and the solution to (6.29) is

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\big(\boldsymbol{x}_0 - \boldsymbol{x}_p(t_0)\big) + \boldsymbol{x}_p(t)\,.$$

To get some insight of solving (6.29), let us first assume that $\boldsymbol{P}(t) = \boldsymbol{A}$ is a time-independent matrix. In such a case,

$$e^{-t\boldsymbol{A}}\boldsymbol{x}' = e^{-t\boldsymbol{A}}\big(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}(t)\big) \quad \text{or} \quad e^{-t\boldsymbol{A}}(\boldsymbol{x}' - \boldsymbol{A}\boldsymbol{x}) = e^{-t\boldsymbol{A}}\boldsymbol{g}(t)\,.$$

Since $\dfrac{d}{dt}e^{-t\boldsymbol{A}} = -\boldsymbol{A}e^{-t\boldsymbol{A}} = -e^{-t\boldsymbol{A}}\boldsymbol{A}$, the equality above implies that

$$\big(e^{-t\boldsymbol{A}}\boldsymbol{x}\big)' = e^{-t\boldsymbol{A}}\boldsymbol{g}(t) \quad \Rightarrow \quad e^{-t\boldsymbol{A}}\boldsymbol{x}(t) - e^{-t_0\boldsymbol{A}}\boldsymbol{x}(t_0) = \int_{t_0}^{t} \boldsymbol{e}^{-s\boldsymbol{A}}\boldsymbol{g}(s)\,ds\,.$$

Therefore, the solution to (6.29) is

$$\boldsymbol{x}(t) = e^{t\boldsymbol{A}}e^{-t_0\boldsymbol{A}}\boldsymbol{x}_0 + \int_{t_0}^{t} e^{t\boldsymbol{A}}e^{-s\boldsymbol{A}}\boldsymbol{g}(s)\,ds\,.$$

Using fundamental matrices $\boldsymbol{\Psi}$ of system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, we have the following similar result.

Theorem 6.29. *Let $\boldsymbol{\Psi}(t)$ be a fundamental matrix of system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, and $\boldsymbol{\varphi}(t)$ be the solution to the non-homogeneous linear system*

$$\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)\,, \tag{6.30a}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0\,. \tag{6.30b}$$

Then $\boldsymbol{\varphi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}_0 + \displaystyle\int_{t_0}^{t} \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds$.

Proof. We directly check that the solution $\boldsymbol{\varphi}$ given above satisfies (6.30). It holds trivially that $\boldsymbol{\varphi}(t_0) = \boldsymbol{x}_0$, so it suffices to show the validity of (6.30a) with $\boldsymbol{\varphi}$ replacing $\boldsymbol{x}$.

Differentiating $\boldsymbol{\varphi}$ and using (6.25), we find that

$$\begin{aligned}\boldsymbol{\varphi}'(t) &= \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}_0 + \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t)^{-1}\boldsymbol{g}(t) + \int_{t_0}^{t} \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds \\ &= \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t)^{-1}\Big(\boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds\Big) + \boldsymbol{g}(t) \\ &= \boldsymbol{P}(t)\boldsymbol{\varphi}(t) + \boldsymbol{g}(t)\end{aligned}$$

which shows that $\boldsymbol{\varphi}$ satisfies (6.30a). □

• Another point of view - variation of parameters: Let $\boldsymbol{\Psi}$ be a fundamental matrix of $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$. We look for a particular solution to $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$. By the method of variation of parameters we can assume that a particular solution can be expressed as

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{u}(t)$$

for some vector-valued function $\boldsymbol{u}$. Since $\boldsymbol{x}$ is a solution, we must have

$$\boldsymbol{\Psi}'(t)\boldsymbol{u}(t) + \boldsymbol{\Psi}(t)\boldsymbol{u}'(t) = \boldsymbol{P}(t)\boldsymbol{\Psi}(t)\boldsymbol{u}(t) + \boldsymbol{g}(t)\,.$$

Since $\boldsymbol{\Psi}' = \boldsymbol{P}(t)\boldsymbol{\Psi}$, we obtain that $\boldsymbol{u}$ satisfies

$$\boldsymbol{u}'(t) = \boldsymbol{\Psi}(t)^{-1}\boldsymbol{g}(t)\,. \tag{6.31}$$

Therefore, we can choose $\boldsymbol{u}(t) = \displaystyle\int \boldsymbol{\Psi}(t)^{-1}\boldsymbol{g}(t)\,dt$ and a particular solution to $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$ is given by

$$\boldsymbol{x}_p(t) = \boldsymbol{\Psi}(t)\Big(\int \boldsymbol{\Psi}(t)^{-1}\boldsymbol{g}(t)\,dt\Big)\,. \tag{6.32}$$

On the other hand, (6.31) implies that $\boldsymbol{u}(t) = \displaystyle\int_{t_0}^{t} \boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds + \boldsymbol{u}(t_0)$, where $\boldsymbol{u}(t_0)$ is the value of $\boldsymbol{u}$ at the initial time given by $\boldsymbol{u}(t_0) = \boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}(t_0)$; thus the solution to $x' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is

$$\begin{aligned}\boldsymbol{x}(t) &= \boldsymbol{\Psi}(t)\Big(\int_{t_0}^{t} \boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds + \boldsymbol{u}(t_0)\Big) \\ &= \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(t_0)^{-1}\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{g}(s)\,ds\,.\end{aligned}$$

Example 6.30. Let $\boldsymbol{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ and $\boldsymbol{g}(t) = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$. Find a particular solution of $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}(t)$.

We first find the Jordan decomposition of $\boldsymbol{A}$. The characteristic equation of $\boldsymbol{A}$ is $(-2-r)^2 - 1 = 0$ which implies that $\lambda = -1$ and $\lambda = -3$ are eigenvalues of $\boldsymbol{A}$. The corresponding eigenvectors are $(1,1)^{\mathrm{T}}$ and $(1,-1)^{\mathrm{T}}$; thus

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\mathrm{T}};$$

thus

$$e^{t\boldsymbol{A}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\mathrm{T}}.$$

The general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

1. To obtain a particular solution, we can use (6.32) and find that

$$\begin{aligned} \boldsymbol{x}_p(t) &= \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \int \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \int \begin{bmatrix} e^{t} & e^{t} \\ e^{3t} & -e^{3t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \int \begin{bmatrix} 2 + 3te^{t} \\ 2e^{2t} - 3te^{3t} \end{bmatrix} dt\,. \end{aligned}$$

Since $\displaystyle\int te^{\lambda t}\,dt = \frac{t}{\lambda}e^{\lambda t} - \frac{1}{\lambda^2}e^{\lambda t}$, we obtain that

$$\boldsymbol{x}_p(t) = \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} 2t + 3(te^{t} - e^{t}) \\ e^{2t} - (te^{3t} - \frac{1}{3}e^{3t}) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2te^{-t} + 3(t-1) + e^{-t} - (t - \frac{1}{3}) \\ 2te^{-t} + 3(t-1) - e^{-t} + (t - \frac{1}{3}) \end{bmatrix}$$

2. Without memorizing the formula (6.32) for a particular solution, we can use the method of variation of parameters by assuming that

$$\boldsymbol{x}_p(t) = C_1(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for some scalar functions C_1, C_2. Then the equation $\boldsymbol{x}_p' = \boldsymbol{A}\boldsymbol{x}_p + \boldsymbol{g}(t)$ implies that

$$\begin{aligned} &C_1'(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - C_1(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2'(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 3C_2(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &\quad = -C_1(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3C_2(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}. \end{aligned}$$

As a consequence

$$C_1'(t)e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2'(t)e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

which implies that

$$\begin{bmatrix} C_1'(t) \\ C_2'(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

The computation above (in 1) can be used to conclude that

$$C_1(t) = 2t + 3(te^{t} - e^{t}) \qquad \text{and} \qquad C_2(t) = e^{2t} - \left(te^{3t} - \frac{1}{3}e^{3t}\right);$$

thus a particular solution is given by

$$\boldsymbol{x}_p(t) = \left[2t + 3(te^{t} - e^{t})\right]e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left[e^{2t} - \left(te^{3t} - \frac{1}{3}e^{3t}\right)\right]e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

6.6 Numerical Methods

So far we only talk about how to find a solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}(t)$ for constant matrix $\boldsymbol{A}$. In general, it is very hard to compute (by hand) the general solution to $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ even if $\boldsymbol{F}(t, \boldsymbol{x}) = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t)$. In this section, we focus on solving the general system (6.8) numerically.

In the following discussion, we do not specify the size of the sysmte $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$; thus n no longer denote the length of the vector $\boldsymbol{x}$.

Definition 6.31 (Informal definition)**.** A numerical method of solving the ODE $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is an iterative scheme which, when the ***step size*** $h > 0$ is given, generates a unique sequence of vectors $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N\}$ (for some N which in general depends on h) such that the piecewise linear function $\boldsymbol{\varphi}$ satisfying

$$\boldsymbol{\varphi}(t) = \frac{\boldsymbol{x}_{n+1} - \boldsymbol{x}_n}{h}(t - t_n) + \boldsymbol{x}_n \quad \forall\, t \in [t_n, t_{n+1}] \text{ and } n \in \{0, 1, \cdots, N-1\},$$

where $t_n = t_0 + nh$, resembles the solution to $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ with initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ in the time interval $[t_0, t_N]$. The function $\boldsymbol{\varphi}$ is called the ***numerical solution*** generated by this numerical method with step size h.

A numerical method of solving the ODE $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ is called a ***k-step method*** if it requires $\boldsymbol{x}_n, \boldsymbol{x}_{n+1}, \cdots, \boldsymbol{x}_{n+k-1}$ to determine $\boldsymbol{x}_{n+k}$ for all $n \in \{0, \cdots, N-k\}$. A numerical method of solving the ODE $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ is said to be ***explicit*** if it does not require "nonlinear procedures" to obtain some $\boldsymbol{x}_n$'s, and is said to be ***implicit*** if it is not explicit.

Example 6.32. The forward Euler method of solving the ordinary differential equations $\boldsymbol{y}' = \boldsymbol{F}(t, \boldsymbol{y})$ is an explicit one-step method given by

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + h\boldsymbol{F}(t_{n-1}, \boldsymbol{x}_{n-1}) \qquad \forall\, n \in \{1, 2, \cdots, N\},$$

while the backward Euler method is an implicit one-step method given by

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + h\boldsymbol{F}(t_n, \boldsymbol{x}_n) \qquad \forall\, n \in \{1, 2, \cdots, N\}.$$

Example 6.33. The Runge-Kutta method involves a weighted average of values of $\mathbf{F}(t, \boldsymbol{x})$ at different points in the interval $t_n \leqslant t \leqslant t_{n+1}$, and is given by

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + h\Big(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\Big), \tag{6.33}$$

where

$$\begin{aligned} k_{n1} &= \boldsymbol{F}(t_n, \boldsymbol{x}_n), & k_{n2} &= \boldsymbol{F}(t_n + \tfrac{1}{2}h, \boldsymbol{x}_n + \tfrac{1}{2}hk_{n1}), \\ k_{n3} &= \boldsymbol{F}(t_n + \tfrac{1}{2}h, \boldsymbol{x}_n + \tfrac{1}{2}hk_{n2}), & k_{n4} &= \boldsymbol{F}(t_n + h, \boldsymbol{x}_n + hk_{n3}). \end{aligned}$$

We note that the Runge-Kutta method is a one-step explicit method.

In this lecture we only consider explicit method.

Remark 6.34. A one-step explicit method is often (but not always) given in the form

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + h\Phi(t_n, \boldsymbol{x}_n)$$

for some function Φ, while a k-step explicit method is often (but not always) given in the form

$$\begin{aligned} \boldsymbol{x}_{n+1} = &\alpha_1 \boldsymbol{x}_n + \alpha_2 \boldsymbol{x}_{n-1} + \cdots + \alpha_k \boldsymbol{x}_{n-k+1} \\ &+h\Big[\beta_1 f(t_n, \boldsymbol{x}_n) + \beta_2 f(t_{n-1}, \boldsymbol{x}_{n-1}) + \cdots + \beta_k f(t_{n-k+1}, \boldsymbol{x}_{n-k+1})\Big] . \end{aligned} \tag{6.34}$$

There are three fundamental sources of error of a numerical solution:

1. The iterative scheme used to produce the sequence $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N\}$ is an approximate one. In other words, at each step the numerical method does not produce the correct value of the solution at the next time step. This relates to the local/global truncation error.

2. The input data used in the iterative scheme are only approximations to the actual values of the solution at each t_k. For example, one should use $\boldsymbol{x}(t_k)$ to generate $\boldsymbol{x}_{k+1}$ but we are forced to start with $\boldsymbol{x}_k$. This relates to the global truncation error.

3. The precision of calculations of the computer is finite. In other words, at each step only a finite number of digits can be retained. This relates to the round-off error (or machine error).

Definition 6.35. Let $\boldsymbol{\varphi}$ be a numerical solution obtained by a specific numerical method (with step size $h > 0$ fixed) of solving ODE $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. At each time step t_n,

1. the ***global truncation error*** (associated with this numerical method) is the number $\boldsymbol{E}_n(h) = \boldsymbol{x}(t_n) - \boldsymbol{\varphi}(t_n)$;

2. the ***local truncation error*** (associated with this numerical method) is the number $\boldsymbol{\tau}_n(h) = \dfrac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}_{n+1}}{h}$, where $\boldsymbol{x}(\cdot)$ is the exact solution and $\boldsymbol{x}_{n+1}$ is obtained according to the iterative scheme with $\boldsymbol{x}_j = \boldsymbol{x}(t_j)$ for all $j \in \{0, 1, \cdots, n\}$.

3. the ***round-off error*** or ***machine error*** (associated with this numerical method) is the number $\boldsymbol{R}_n = \boldsymbol{\varphi}(t_n) - \boldsymbol{X}_n$, where $\boldsymbol{X}_n$ is the actual value computed from the numerical method.

In other words, the local truncation error measures the accuracy of the numerical method for each time step, while the global truncation error measure the errors accumulated from the beginning of this iterative scheme.

Definition 6.36. A numerical method is said to be ***consistent*** if

$$\lim_{h \to 0} \max_{0 \leqslant n \leqslant N-1} |\boldsymbol{\tau}_n(h)| = 0 ,$$

where $\boldsymbol{\tau}_n(h)$ is the local truncation error associated with the numerical method with step size h.

Example 6.37. Consider the forward Euler method of solving $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$. If $\boldsymbol{x}(t)$ is the solution to $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ with initial data $\boldsymbol{x}(t_n)$, then the Euler method provides an approximated value of $\boldsymbol{x}(t_{n+1})$ given by

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}(t_n) + h\boldsymbol{F}\big(t_n, \boldsymbol{x}(t_n)\big)\,.$$

The local truncation error is then computed by

$$\begin{aligned}\tau_n(h) &= \frac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}_{n+1}}{h} = \frac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}(t_n) - h\boldsymbol{F}\big(t_n, \boldsymbol{x}(t_n)\big)}{h} = \frac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}(t_n)}{h} - \boldsymbol{F}\big(t_n, \boldsymbol{x}(t_n)\big)\\ &= \frac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}(t_n)}{h} - \boldsymbol{x}'(t_n)\,.\end{aligned}$$

In other words, the local truncation $\tau_n(h)$ (of the forward Euler method) measures the difference between the real derivative and the "discrete derivative" (which allows us to design the numerical scheme).

Now consider the backward Euler method. Similar computation shows that the local truncation error $\tau_n(h)$ associated with the backward Euler method is given by

$$\tau_n(h) = \frac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}(t_n)}{h} - \boldsymbol{x}'(t_{n+1})\,.$$

Example 6.38. Consider the Runge-Kutta method given in Example 6.33. The local truncation error $\tau_n(h)$ is given by

$$\begin{aligned}\frac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}_{n+1}}{h} &= \frac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}(t_n) + h\Big(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\Big)}{h}\\ &= \frac{\boldsymbol{x}(t_{n+1}) - \boldsymbol{x}(t_n)}{h} - \frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\,,\end{aligned}$$

where

$$\begin{aligned}k_{n1} &= \boldsymbol{F}\big(t_n, \boldsymbol{x}(t_n)\big)\,, & k_{n2} &= \boldsymbol{F}\big(t_n + \tfrac{1}{2}h, \boldsymbol{x}(t_n) + \tfrac{1}{2}hk_{n1}\big)\,,\\ k_{n3} &= \boldsymbol{F}\big(t_n + \tfrac{1}{2}h, \boldsymbol{x}(t_n) + \tfrac{1}{2}hk_{n2}\big)\,, & k_{n4} &= \boldsymbol{F}\big(t_n + h, \boldsymbol{x}(t_n) + hk_{n3}\big)\,.\end{aligned}$$

Assume that $\boldsymbol{F}$ is a scalar function (so that it is easy for the purpose of demonstration) and $\boldsymbol{F}$ is of class $\mathscr{C}^2$ (that is, $\boldsymbol{F}$ is twice continuously differentiable). Then $k_{n1} = \boldsymbol{x}'(t_n)$, and

$$\begin{aligned}k_{n2} &= \boldsymbol{F}\big(t_n, \boldsymbol{x}(t_n)\big) + \boldsymbol{F}_t\big(t_n, \boldsymbol{x}(t_n)\big)\frac{h}{2} + \boldsymbol{F}_{\boldsymbol{x}}\big(t_n, \boldsymbol{x}(t_n)\big)\frac{hk_{n1}}{2} + \mathcal{O}(h^2)\\ &= \boldsymbol{x}'(t_n) + \frac{h}{2}\big[\boldsymbol{F}_t\big(t_n, \boldsymbol{x}(t_n)\big) + \boldsymbol{F}_{\boldsymbol{x}}\big(t_n, \boldsymbol{x}(t_n)\big)k_{n1}\big] + \mathcal{O}(h^2)\,,\\ k_{n3} &= \boldsymbol{F}\big(t_n, \boldsymbol{x}(t_n)\big) + \boldsymbol{F}_t\big(t_n, \boldsymbol{x}(t_n)\big)\frac{h}{2} + \boldsymbol{F}_{\boldsymbol{x}}\big(t_n, \boldsymbol{x}(t_n)\big)\frac{hk_{n2}}{2} + \mathcal{O}(h^2)\\ &= \boldsymbol{x}'(t_n) + \frac{h}{2}\big[\boldsymbol{F}_t\big(t_n, \boldsymbol{x}(t_n)\big) + \boldsymbol{F}_{\boldsymbol{x}}\big(t_n, \boldsymbol{x}(t_n)\big)k_{n2}\big] + \mathcal{O}(h^2)\,,\\ k_{n4} &= \boldsymbol{F}\big(t_n, \boldsymbol{x}(t_n)\big) + \boldsymbol{F}_t\big(t_n, \boldsymbol{x}(t_n)\big)h + \boldsymbol{F}_{\boldsymbol{x}}\big(t_n, \boldsymbol{x}(t_n)\big)hk_{n3} + \mathcal{O}(h^2)\\ &= \boldsymbol{x}'(t_n) + h\big[\boldsymbol{F}_t\big(t_n, \boldsymbol{x}(t_n)\big) + \boldsymbol{F}_{\boldsymbol{x}}\big(t_n, \boldsymbol{x}(t_n)\big)k_{n3}\big] + \mathcal{O}(h^2)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{k_{n1}+2k_{n2}+2k_{n3}+k_{n4}}{6} &= \boldsymbol{x}'(t_n)+\frac{h}{2}\boldsymbol{F}_t\big(t_n,\boldsymbol{x}(t_n)\big)+\frac{h}{6}\boldsymbol{F}_{\boldsymbol{x}}\big(t_n,\boldsymbol{x}(t_n)\big)\big[\boldsymbol{x}'(t_n)+k_{n2}+k_{n3}\big]+\mathcal{O}(h^2)\\ &= \boldsymbol{x}'(t_n)+\frac{h}{2}\big[\boldsymbol{F}_t\big(t_n,x(t_n)\big)+\boldsymbol{F}_{\boldsymbol{x}}\big(t_n,x(t_n)\big)\boldsymbol{x}'(t_n)\big]+\mathcal{O}(h^2)\\ &= \boldsymbol{x}'(t_n)+\frac{h}{2}\boldsymbol{x}''(t_n)+\mathcal{O}(h^2)\end{aligned}$$

which implies that

$$\begin{aligned}\tau_n(h) &= \frac{\boldsymbol{x}(t_{n+1})-\boldsymbol{x}(t_n)}{h}-\frac{k_{n1}+2k_{n2}+2k_{n3}+k_{n4}}{6}\\ &= \frac{\boldsymbol{x}(t_{n+1})-\boldsymbol{x}(t_n)}{h}-\boldsymbol{x}'(t_n)-\frac{h}{2}\boldsymbol{x}''(t_n)+\mathcal{O}(h^2)\,.\end{aligned}$$

Since $\boldsymbol{F}$ is of class $\mathscr{C}^2$, $\boldsymbol{x}$ is of class $\mathscr{C}^3$; thus the Taylor theorem implies that

$$\boldsymbol{x}(t_{n+1})=\boldsymbol{x}(t_n)+h\boldsymbol{x}'(t_n)+\frac{h^2}{2}\boldsymbol{x}'(t_n)+\mathcal{O}(h^3)\,.$$

As a consequence,

$$\tau_n(h)=\frac{h\boldsymbol{x}'(t_n)+\frac{h^2}{2}\boldsymbol{x}'(t_n)+\mathcal{O}(h^3)}{h}-\boldsymbol{x}'(t_n)-\frac{h}{2}\boldsymbol{x}''(t_n)+\mathcal{O}(h^2)=\mathcal{O}(h^2)\,.$$

Remark 6.39. If one assume that $\boldsymbol{F}$ is of class $\mathscr{C}^4$, then the Runge-Kutta method provides numerical solutions with local truncation error of order 4; that is, $\tau_n(h)=\mathcal{O}(h^4)$.

• **Further look at the local truncation error and the consistency**: Now we take a look at what the local truncation error for an one-step numerical scheme

$$\boldsymbol{x}_{n+1}=\boldsymbol{x}_n+h\Phi(h,t_n,\boldsymbol{x}_n) \tag{6.35}$$

really means. We remark that here $\Phi(h,t_n,\boldsymbol{x}_n)$ can be viewed as a way to approximate the derivative in the time interval $[t_n,t_{n+1}]$. Moreover, both Euler method and Runge-Kutta scheme are one-step methods under this definition.

Before continuing the discussion, let us use the convention that for $h>0$ and fixed $T>0$, we define $N=\big[\frac{T}{h}\big]$ and the numerical solution $\boldsymbol{\varphi}$ generated by the one-step scheme (6.35) on the time interval $\big[t_N,t_0+T\big]$ (on which $\boldsymbol{\varphi}$ is not defined) has value $\boldsymbol{\varphi}(t_N)$.

Suppose that there exists a unique solution $\boldsymbol{x}$ to (6.8) on the time interval $[t_0,t_0+T]$. By definition of the local truncation error,

$$\begin{aligned}\tau_n(h)=\frac{\boldsymbol{x}(t_{n+1})-x_{n+1}}{h} &= \frac{\boldsymbol{x}(t_{n+1})-\boldsymbol{x}(t_n)}{h}-\Phi\big(h,t_n,\boldsymbol{x}(t_n)\big)\\ &= \frac{\boldsymbol{x}(t_{n+1})-\boldsymbol{x}(t_n)}{h}-\boldsymbol{x}'(t_n)+\boldsymbol{F}\big(t_n,\boldsymbol{x}(t_n)\big)+\Phi\big(h,t_n,\boldsymbol{x}(t_n)\big)\,.\end{aligned}$$

If $\boldsymbol{F}$ is continuous on $K\equiv[t_0,t_0+T]\times\Big[\min\limits_{t\in[t_0,t_0+T]}\boldsymbol{x}(t),\max\limits_{t\in[t_0,t_0+T]}\boldsymbol{x}(t)\Big]$, then $\boldsymbol{x}$ must be $\mathscr{C}^1$ which implies that $\boldsymbol{x}'$ is uniformly continuous on $[t_0, t_0+T]$. Therefore, the mean value theorem implies that

$$\lim_{h\to 0}\max_{1\leqslant n\leqslant N}\left|\frac{\boldsymbol{x}(t_{n+1})-\boldsymbol{x}(t_n)}{h}-\boldsymbol{x}'(t_n)\right|=0\,;$$

thus

$$\lim_{h\to 0}\max_{1\leqslant n\leqslant N}|\tau_n(h)|=\lim_{h\to 0}\max_{1\leqslant n\leqslant N}\left|\boldsymbol{F}\big(t_n,\boldsymbol{x}(t_n)\big)-\Phi\big(h,t_n,\boldsymbol{x}(t_n)\big)\right|. \tag{6.36}$$

If we further assume the uniform continuity of Φ on its variables, then (6.36) further implies that

$$\lim_{h\to 0}\max_{1\leqslant n\leqslant N}|\tau_n(h)|=\lim_{h\to 0}\sup_{t\in[t_0,t_0+T]}\left|\boldsymbol{F}\big(t,\boldsymbol{x}(t)\big)-\Phi\big(h,t,\boldsymbol{x}(t)\big)\right| \tag{6.37}$$

Therefore, assuming that $\boldsymbol{F}$ is Lipchitz continuous (which guarantees the existence and uniqueness of the solution $\boldsymbol{x}$ to (6.8) by Theorem 6.5) and Φ is uniformly continuous, then the consistency of the one-step numerical scheme (6.35) is equivalent to that $\Phi\big(h,\cdot,\boldsymbol{x}(\cdot)\big)$ converges uniformly to $\boldsymbol{F}\big(\cdot,\boldsymbol{x}(\cdot)\big)$ as $h\to 0$ (that is, (6.37)).

Remark 6.40. To see (6.36), let $\varepsilon>0$ and $t\in[t_0,t_0+T]$ be given. Since $\boldsymbol{x}'$ is continuous on $[t_0,t_0+T]$, for some $M>0$ we have $\big|\boldsymbol{x}'(t)\big|\leqslant M$ for all $t\in[t_0,t_0+T]$. Then

1. the uniform continuity of $\boldsymbol{F}$ provides a $\delta_1>0$ such that

$$\big|\boldsymbol{F}(t,\boldsymbol{y})-\boldsymbol{F}(s,\boldsymbol{z})\big|<\frac{\varepsilon}{2}\quad\text{whenever } (t,\boldsymbol{y}),(s,\boldsymbol{z})\in K \text{ satisfying } |t-s|^2+|\boldsymbol{y}-\boldsymbol{z}|^2<\delta_1^2\,;$$

2. the uniform continuity of Φ provides a $\delta_2>0$ such that

$$\big|\Phi(h,t,\boldsymbol{y})-\Phi(h,s,\boldsymbol{z})\big|<\frac{\varepsilon}{2}\quad\text{whenever } |t-s|^2+|\boldsymbol{y}-\boldsymbol{z}|^2<\delta_2^2\,.$$

Let $\delta=\dfrac{\min\{\delta_1,\delta_2\}}{M^2+1}$. If $0<h<\delta$, there exists t_n such that $|t_n-t|<\delta$. Therefore, $\big|\big(t_n,\boldsymbol{x}(t_n)\big)-\big(t,\boldsymbol{x}(t)\big)\big|<\min\{\delta_1,\delta_2\}$; thus

$$\big|\boldsymbol{F}\big(t_n,\boldsymbol{x}(t_n)\big)-\boldsymbol{F}\big(t,\boldsymbol{x}(t)\big)\big|+\big|\Phi\big(h,t_n,\boldsymbol{x}(t_n)\big)-\Phi\big(h,t,\boldsymbol{x}(t)\big)\big|<\varepsilon\,.$$

6.6.1 Convergence of Bounded Consistent Schemes

We first consider the convergence of numerical solutions obtained from a bounded (which is defined in Theorem 6.44) numerical scheme as the step size approaches zero.

Definition 6.41. Let $a,b\in\mathbb{R}$ and $a<b$. A family $\mathscr{F}$ of functions in $\mathscr{C}([a,b];\mathbb{R}^n)$ (which means for each $f\in\mathscr{F}$, $f:[a,b]\to\mathbb{R}^n$ is a continuous vector-valued function) is said to be

1. ***uniformly bounded*** if

$$\exists\, M>0 \ni |f(t)|\leqslant M\quad \forall\, t\in[a,b]\quad\text{and}\quad f\in\mathscr{F}\,;$$

2. ***equi-continuous*** if

$$\forall\,\varepsilon>0,\exists\,\delta>0\ni\big|f(t)-f(s)\big|<\varepsilon\quad\text{whenever } |t-s|<\delta,\ t,s\in[a,b],\text{ and } k\in\mathbb{N}\,.$$

Now we introduce the Arzelà-Ascoli Theorem which can be applied to extract a uniformly convergent subsequence from a sequence of continuous functions as long as those functions are equi-continuous and uniformly bounded.

Theorem 6.42 (Arzelà-Ascoli)**.** *Let $a, b \in \mathbb{R}$ with $a < b$, and $\{f_k\}_{k=1}^{\infty} \subseteq \mathscr{C}([a,b];\mathbb{R}^n)$ be a uniformly bounded, equi-continuous sequence of functions. Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ which converges uniformly (to some function $f \in \mathscr{C}([a,b];\mathbb{R}^n)$).*

Remark 6.43. The uniform convergence of the sequence $\{f_k\}_{k=1}^{\infty}$ to f on $[a,b]$ means that

$$\lim_{k\to\infty} \sup_{x\in[a,b]} |f_k(x) - f(x)| = 0\,.$$

Theorem 6.44. *For each $h > 0$, let $\boldsymbol{\varphi}_h : [t_0, t_0+T] \to \mathbb{R}^n$ be the numerical solution generated by the one-step scheme* (6.35) *for some functions Φ. If Φ is bounded near $h = 0$; that is, there exists $\delta > 0$ and $M > 0$ such that $\big|\Phi(h,t,\boldsymbol{x})\big| \leqslant M$ for all $(h,t,\boldsymbol{x}) \in (0,\delta] \times [t_0,t_0+T] \times \mathbb{R}^n$, then every subsequence $\{\boldsymbol{\varphi}_{h_j}\}_{j=1}^{\infty}$ of $\{\boldsymbol{\varphi}_h\}_{h>0}$ possesses a uniformly convergent subsequence $\{\boldsymbol{\varphi}_{h_{j_\ell}}\}_{\ell=1}^{\infty}$; that is, for some continuous function $\boldsymbol{\varphi} : [t_0,t_0+T] \to \mathbb{R}^n$, we have*

$$\lim_{\ell\to\infty} \sup_{t\in[t_0,t_0+T]} \big|\boldsymbol{\varphi}_{h_{j_\ell}}(t) - \boldsymbol{\varphi}(t)\big| = 0\,.$$

Proof. By Arzelà-Ascoli Theorem, it suffices to prove that the family of functions $\{\boldsymbol{\varphi}_h\}_{h>0}$ is uniformly bounded and equi-continuous.

First, since Φ is bounded by M, the numerical scheme implies that

$$\begin{aligned}\big|\boldsymbol{\varphi}_h(t_n)\big| &\leqslant \big|\boldsymbol{\varphi}_h(t_{n-1})\big| + hM \leqslant \big|\boldsymbol{\varphi}_h(t_{n-2})\big| + 2hM \leqslant \cdots \\ &\leqslant \big|\boldsymbol{\varphi}_h(0)\big| + nhM \leqslant |x_0| + MT \qquad \forall\, n \in \Big\{1,\cdots,\Big[\frac{T}{h}\Big]\Big\}\,.\end{aligned}$$

Since $\boldsymbol{\varphi}_h$ are piecewise linear for all $h > 0$, we find that $\big|\boldsymbol{\varphi}_h(t)\big| \leqslant |x_0|e^{MT}$ for all $t \in [t_0,t_0+T]$ and for all $h > 0$. Therefore, $\{\boldsymbol{\varphi}_h\}_{h>0}$ is a uniformly bounded family of continuous functions.

On the other hand, by the boundedness of Φ again, we find that

$$\Big|\frac{\boldsymbol{\varphi}_h(t) - \boldsymbol{\varphi}_h(s)}{t-s}\Big| \leqslant M \qquad \forall\, t,s \in [t_{n-1},t_n] \text{ for some } n \in \Big\{1,\cdots,\Big[\frac{T}{h}\Big]\Big\}.$$

Therefore, by the fact that

$$\Big|\frac{\boldsymbol{\varphi}_h(t) - \boldsymbol{\varphi}_h(s)}{t-s}\Big| \leqslant \max\Big\{\Big|\frac{\boldsymbol{\varphi}_h(t) - \boldsymbol{\varphi}_h(r)}{t-r}\Big|, \Big|\frac{\boldsymbol{\varphi}_h(r) - \boldsymbol{\varphi}_h(s)}{r-s}\Big|\Big\} \qquad \forall\, s \leqslant r \leqslant t\,,$$

we find that $\{\varphi_h\}_{h>0}$ is uniformly Lipschitz (with Lipschitz constant M) which implies that $\{\boldsymbol{\varphi}_h\}_{h>0}$ is an equi-continuous family of continuous functions. □

Therefore, every bounded numerical scheme produces a limit function which is a candidate of the exact solution. Next, we consider the convergence (to the exact solution) of numerical solutions generated by a bounded consistent numerical scheme.

Theorem 6.45. *Suppose that* $\boldsymbol{F} : [t_0, t_0 + T] \times \mathbb{R}^n$ *be uniformly continuous such that the system* (6.8) *has a unique solution* $\boldsymbol{x} : [t_0, t_0 + T] \to \mathbb{R}^n$*. For each* $h > 0$*, let* $\boldsymbol{\varphi}_h : [t_0, t_0 + T] \to \mathbb{R}^n$ *be the numerical solution generated by the one-step scheme* (6.35) *for some functions* Φ*. If for some* $\delta > 0$*,* Φ *is bounded uniformly continuous on* $(0, \delta] \times [t_0, t_0 + T] \times \mathbb{R}^n$*, and* $\Phi(h, \cdot, \cdot)$ *converges to* $\boldsymbol{F}$ *uniformly on* $[t_0, t_0 + T] \times \mathbb{R}^n$ *as* $h \to 0$*, then the sequence* $\{\boldsymbol{\varphi}_h\}_{h>0}$ *converges uniformly to the exact solution* $\boldsymbol{x}$ *to* (6.8).

Proof. If suffices to show that if the sequence $\{\boldsymbol{\varphi}_{h_j}\}_{j=1}^{\infty}$ converges to some function $\boldsymbol{\varphi}$ uniformly on $[t_0, t_0 + T]$, then $\boldsymbol{\varphi}$ must be the solution $\boldsymbol{x}$ to (6.8).

Let $\varepsilon > 0$ be given. Then the uniform continuity of Φ implies that there exists $\delta_1 > 0$ such that if $|\boldsymbol{y} - \boldsymbol{z}| < \delta_1$, then

$$\sup_{h>0, k\in\{0,\cdots,\ell\}} \big|\Phi\big(h, kh, \boldsymbol{y}\big) - \Phi\big(h, kh, \boldsymbol{z}\big)\big| < \frac{\varepsilon}{3T}\,.$$

The uniform convergence of $\Phi(h, \cdot, \cdot)$ to $\boldsymbol{F}$ as $h \to 0$ implies that there exists $\delta_2 > 0$ such that if $0 < h < \delta_2$,

$$\sup_{(t,\boldsymbol{y})\in(0,\delta]\times\mathbb{R}^n} \big|\boldsymbol{F}\big(t, \boldsymbol{y}\big) - \Phi\big(h, t, \boldsymbol{y}\big)\big| < \frac{\varepsilon}{3T}\,.$$

Moreover, since $\boldsymbol{F}\big(\cdot, \boldsymbol{\varphi}(\cdot)\big)$ is continuous on $[t_0, t_0 + T]$, it is Riemann integrable over $[t_0, t_0 + T]$; thus there exists $\delta_3 > 0$ such that if $0 < h < \delta_3$,

$$\Big|\int_{t_0}^{t} \boldsymbol{F}\big(s, \boldsymbol{\varphi}(s)\big)\,ds - \sum_{k=0}^{\ell-1} \boldsymbol{F}\big(t_0 + kh, \boldsymbol{\varphi}(t_0 + kh)\big)h\Big| < \frac{\varepsilon}{3}\,.$$

Let $t \in [t_0, t_0 + T]$ be given. For $h_j > 0$, we define $\ell_j = \big[\frac{t}{h_j}\big]$. Then $\boldsymbol{\varphi}_{h_j}$ satisfies that

$$\begin{aligned}
\boldsymbol{\varphi}_{h_j}(t_0 + h_j) - \boldsymbol{\varphi}_{h_j}(t_0) &= h_j\Phi\big(h_j, t_0, \boldsymbol{\varphi}_{h_j}(t_0)\big)\,,\\
\boldsymbol{\varphi}_{h_j}(t_0 + 2h_j) - \boldsymbol{\varphi}_{h_j}(t_0 + h_j) &= h_j\Phi\big(h_j, t_0 + h_j, \boldsymbol{\varphi}_{h_j}(t_0 + h_j)\big)\,,\\
\vdots \qquad &= \qquad \vdots\\
\boldsymbol{\varphi}_{h_j}\big(t_{\ell_j} - h_j\big) - \boldsymbol{\varphi}_{h_j}\big(t_{\ell_j} - 2h_j\big) &= h_j\Phi\big(h_j, t_0 + (\ell_j - 2)h_j, \boldsymbol{\varphi}_{h_j}(t_0 + (\ell_j - 2)h_j)\big)\,,\\
\boldsymbol{\varphi}_{h_j}(t_{\ell_j}) - \boldsymbol{\varphi}_{h_j}(t_{\ell_j} - h_j) &= h_j\Phi\big(h_j, t_0 + (\ell_j - 1)h_j, \boldsymbol{\varphi}_{h_j}(t_0 + (\ell_j - 1)h_j)\big)\,.
\end{aligned}$$

Summing all the equalities, we find that

$$\begin{aligned}
\boldsymbol{\varphi}_{h_j}(t_{\ell_j}) &= \boldsymbol{\varphi}_{h_j}(t_0) + \sum_{k=0}^{\ell_j-1} \Phi\big(h_j, t_0 + kh_j, \boldsymbol{\varphi}_{h_j}(t_0 + kh_j)\big)h_j\\
&= \boldsymbol{x}_0 + \sum_{k=0}^{\ell_j-1} \Phi\big(h_j, t_0 + kh_j, \boldsymbol{\varphi}_{h_j}(t_0 + kh_j)\big)h_j\,.
\end{aligned} \tag{6.38}$$

Now, by the uniform convergence of $\{\boldsymbol{\varphi}_{h_j}\}_{j=1}^{\infty}$ and $h_j \to 0$ as $j \to \infty$, there exists $N > 0$ such that if $j \geqslant N$,

$$\sup_{t\in[0,T]} \big|\boldsymbol{\varphi}_{h_j}(t) - \boldsymbol{\varphi}(t)\big| < \delta_1 \quad \text{and} \quad 0 < h_j < \min\{\delta_2, \delta_3\}\,.$$

Therefore, if $j \geqslant N$, identity (6.38) yields that

$$
\begin{aligned}
&\Big|\boldsymbol{\varphi}_{h_j}(t_{\ell_j}) - \boldsymbol{x}_0 - \int_{t_0}^t \boldsymbol{F}\big(s,\boldsymbol{\varphi}(s)\big)\,ds\Big| \\
&\quad \leqslant \Big|\int_{t_0}^t \boldsymbol{F}\big(s,\boldsymbol{\varphi}(s)\big)\,ds - \sum_{k=0}^{\ell-1} \boldsymbol{F}(t_0+kh_j,\boldsymbol{\varphi}(t_0+kh_j))h_j\Big| \\
&\qquad + \sum_{k=0}^{\ell-1} \big|\boldsymbol{F}(t_0+kh_j,\boldsymbol{\varphi}(t_0+kh_j)) - \Phi\big(h_j,t_0+kh_j,\boldsymbol{\varphi}(t_0+kh_j)\big)\big|h_j \\
&\qquad + \sum_{k=0}^{\ell-1} \big|\Phi\big(h_j,t_0+kh_j,\boldsymbol{\varphi}(t_0+kh_j)\big) - \Phi\big(h_j,t_0+kh_j,\boldsymbol{\varphi}_{h_j}(t_0+kh_j)\big)\big|h_j \\
&\quad \leqslant \frac{\varepsilon}{3} + 2\cdot\frac{\varepsilon}{3T}\sum_{k=0}^{\ell-1} h_j \leqslant \varepsilon\,.
\end{aligned}
$$

Passing to the limit as $j \to \infty$, by the fact that $t_{\ell_j} \to t$ as $j \to 0$ and $\boldsymbol{\varphi}_{h_j}$ converges to $\boldsymbol{\varphi}$ uniformly as $j \to \infty$, we find that

$$
\Big|\boldsymbol{\varphi}(t) - \boldsymbol{x}_0 - \int_{t_0}^t \boldsymbol{F}\big(s,\boldsymbol{\varphi}(s)\big)\,ds\Big| \leqslant \varepsilon\,.
$$

Since $\varepsilon > 0$ is given arbitrarily, we conclude that

$$
\boldsymbol{\varphi}(t) = \boldsymbol{x}_0 + \int_{t_0}^t \boldsymbol{F}\big(s,\boldsymbol{\varphi}(s)\big)\,ds \qquad \forall\, t \in [t_0, t_0+T]\,.
$$

The identity above implies that $\boldsymbol{\varphi}$ is differentiable and $\boldsymbol{\varphi}' = \boldsymbol{F}(t,\boldsymbol{\varphi})$ and $\boldsymbol{\varphi}(t_0) = \boldsymbol{x}_0$. Since $\boldsymbol{x}$ is the unique solution to (6.8), we must have $\boldsymbol{\varphi}(t) = \boldsymbol{x}(t)$. □

Example 6.46. If $\boldsymbol{F} : [t_0, t_0+T] \times \mathbb{R}^n$ is bounded and Lipschitz continuous, then the forward Euler method provides a sequence of numerical solutions $\{\boldsymbol{\varphi}_h\}_{h>0}$ which converges to the exact solution to $\boldsymbol{x}' = \boldsymbol{F}(t,\boldsymbol{x})$.

Example 6.47. Consider the ODE $x' = \sin(x^2)$ with initial data $x(0) = 5$. One can use the following matlab® code

```
T = 1; % the duration of time
h = 0.01; % the step size
N = floor(T/h); % the number of total steps
x = zeros(length(N+1));
x(1) = 5; % assign the initial data

for j=1:1:N
   x(j+1) = x(j) + h*sin(x(j)∧2);
end

t = 0:h:N*h;
plot(t,x);
```

to generate a numerical solution.

6.6.2 The Rate of Convergence

In this sub-section, we focus on how fast a sequence of numerical solutions converges to the exact solution (given that the assumptions in Theorem 6.45 and probably more assumptions hold). To see the rate of convergence, we have to look at the convergence behavior of the global truncation error.

By the mean value theorem (for functions of several variables),

$$\begin{aligned}\boldsymbol{E}_n(h) &= \boldsymbol{x}(t_n) - \boldsymbol{x}_n = \boldsymbol{x}(t_n) - \boldsymbol{x}_{n-1} - h\Phi(h, t_{n-1}, \boldsymbol{x}_{n-1}) \\ &= \boldsymbol{x}(t_n) - \boldsymbol{x}(t_{n-1}) - h\Phi(h, t_{n-1}, \boldsymbol{x}(t_{n-1})) + \boldsymbol{x}(t_{n-1}) - \boldsymbol{x}_{n-1} \\ &\quad + h\big[\Phi(h, t_{n-1}, \boldsymbol{x}(t_{n-1})) - \Phi(h, t_{n-1}, \boldsymbol{x}_{n-1})\big] \\ &= h\boldsymbol{\tau}_{n-1}(h) + \boldsymbol{E}_{n-1}(h) + h(\nabla_{\boldsymbol{x}}\Phi)(h, t_{n-1}, \xi_{n-1})\big[\boldsymbol{x}(t_{n-1}) - \boldsymbol{x}_{n-1}\big] \\ &= h\boldsymbol{\tau}_{n-1}(h) + \boldsymbol{E}_{n-1}(h) + h(\nabla_{\boldsymbol{x}}\Phi)(h, t_{n-1}, \xi_{n-1})\boldsymbol{E}_{n-1}(h)\end{aligned}$$

for some ξ_{n-1} on the line segment joining $\boldsymbol{x}(t_{n-1})$ and $\boldsymbol{x}_{n-1}$. If we assume that $\big|\nabla_{\boldsymbol{x}}\Phi\big|$ is bounded by K, then the equality above implies that

$$|\boldsymbol{E}_n(h)| \leqslant h|\boldsymbol{\tau}_{n-1}(h)| + (1 + hK)|\boldsymbol{E}_{n-1}(h)|\,.$$

Therefore,

$$\begin{aligned}|\boldsymbol{E}_n(h)| &\leqslant h|\boldsymbol{\tau}_{n-1}(h)| + (1 + hK)|\boldsymbol{E}_{n-1}(h)|\,, \\ (1 + hK)|\boldsymbol{E}_{n-1}(h)| &\leqslant h(1 + hK)|\boldsymbol{\tau}_{n-2}(h)| + (1 + hK)^2|\boldsymbol{E}_{n-2}(h)|\,, \\ \vdots \quad &\leqslant \quad \vdots \\ (1 + hK)^{n-1}|\boldsymbol{E}_1(h)| &\leqslant h(1 + hK)^{n-1}|\boldsymbol{\tau}_0(h)| + (1 + hK)^n|\boldsymbol{E}_0(h)|\,.\end{aligned}$$

Summing all the inequalities above, we find that

$$|\boldsymbol{E}_n(h)| \leqslant h\sum_{k=1}^{n}(1 + hK)^{k-1}\big|\boldsymbol{\tau}_{n-k}(h)\big| + (1 + hK)^n|\boldsymbol{E}_0(h)|\,. \tag{6.39}$$

Suppose that the local truncation error satisfies

$$|\boldsymbol{\tau}_n(h)| \leqslant Ah^r \qquad \forall\, n \in \big\{0, 1, \cdots, \big[\tfrac{T}{h}\big] - 1\big\}$$

for some constant A and $r > 0$. Then by the fact $\boldsymbol{E}_0(h) = \boldsymbol{0}$, we conclude that

$$\begin{aligned}|\boldsymbol{E}_n(h)| &\leqslant h\sum_{k=1}^{n}(1 + hK)^{k-1}Ah^r + (1 + hK)^n|\boldsymbol{E}_0(h)| \leqslant h\frac{(1 + hK)^n - 1}{hK}Ah^r \\ &\leqslant \frac{1}{K}\big[(1 + hK)^{\frac{T}{h}} - 1\big]Ah^r \leqslant \frac{1}{K}(e^{KT} - 1)Ah^r\,.\end{aligned}$$

Therefore, we establish the following

Theorem 6.48. *Assume the conditions in Theorem 6.45. If $\nabla_{\boldsymbol{x}}\Phi$ is bounded by K and the local truncation error $\boldsymbol{\tau}_n(h)$ satisfies*

$$|\boldsymbol{\tau}_n(h)| \leqslant Ah^r \qquad \forall\, n \in \big\{0, 1, \cdots, \big[\tfrac{T}{h}\big] - 1\big\} \text{ and } h > 0$$

for some constant $A > 0$, then the global truncation error $\boldsymbol{E}_n(h)$ satisfies

$$|\boldsymbol{E}_n(h)| \leqslant \frac{1}{K}(e^{KT} - 1)h^r \qquad \forall\, h > 0\,.$$

Example 6.49. If $\boldsymbol{F} : [t_0, t_0 + T] \times \mathbb{R}^n$ is bounded and Lipschitz continuous, then the forward Euler method provides a sequence of numerical solutions $\{\boldsymbol{\varphi}_h\}_{h>0}$ which converges to the exact solution to $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ with rate

$$\sup_{t \in [t_0, t_0+T]} \big|\boldsymbol{\varphi}_h(t) - \boldsymbol{x}(t)\big| \leqslant Ch \qquad \forall\, h > 0$$

for some constant $C > 0$. To see this, we first note that Theorem 6.48 implies that there exists $M > 0$ such that

$$\sup_{0 \leqslant n \leqslant [T/h]} \big|\boldsymbol{\varphi}_h(t_n) - \boldsymbol{x}(t_n)\big| \leqslant Mh \qquad \forall\, h > 0\,.$$

Suppose that $|\boldsymbol{F}|$ is bounded by L. For $t \in [t_0, t_0 + T]$, choose $n \in \{0, 1, \cdots, N - 1\}$ such that $t_n \leqslant t \leqslant t_{n+1}$. Then

$$\big|\boldsymbol{\varphi}_h(t) - \boldsymbol{\varphi}_h(t_n)\big| \leqslant \big|\boldsymbol{\varphi}_h(t_{n+1}) - \boldsymbol{\varphi}_h(t_n)\big| = h\big|\boldsymbol{F}\big(t_n, \boldsymbol{\varphi}_h(t_n)\big)\big| \leqslant Lh$$

and the mean value theorem implies that

$$\big|\boldsymbol{x}(t) - \boldsymbol{x}(t_n)\big| \leqslant \big|\boldsymbol{x}'(\xi)\big||t - t_n| = \big|\boldsymbol{F}\big(\xi, \boldsymbol{x}(\xi)\big)\big||t - t_n| \leqslant Lh\,.$$

Therefore,

$$\big|\boldsymbol{x}(t) - \boldsymbol{\varphi}_h(t)\big| \leqslant \big|\boldsymbol{x}(t) - \boldsymbol{x}(t_n)\big| + \big|\boldsymbol{x}(t_n) - \boldsymbol{\varphi}_h(t_n)\big| + \big|\boldsymbol{\varphi}_h(t_n) - \boldsymbol{\varphi}_h(t)\big| \leqslant (M + 2L)h \quad \forall\, h > 0\,.$$

7 The Laplace Transform

7.1 Definition of the Laplace Transform

Definition 7.1 (Integral transform)**.** An ***integral transform*** is a relation between two functions f and F of the form

$$F(s) = \int_\alpha^\beta K(s, t) f(t)\, dt\,, \tag{7.1}$$

where $K(\cdot, \cdot)$ is a given function, called the ***kernel*** of the transformation, and the limits of integration α, β are also given (here α, β could be ∞ and in such cases the integral above is an improper integral). The relation (7.1) transforms function f into another function F called the transformation of f.

Proposition 7.2. *Every integral transform is linear; that is, for all functions f and g (defined on (α, β)) and constant a,*

$$\int_\alpha^\beta K(s, t)\big(af(t) + g(t)\big)\, dt = a \int_\alpha^\beta K(s, t) f(t)\, dt + \int_\alpha^\beta K(s, t) g(t)\, dt\,.$$

Example 7.3. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $\int_{-\infty}^{\infty}|f(x)|\,dx<\infty$. The ***Fourier transform*** of f, denoted by $\mathscr{F}(f)$, is defined by

$$\mathscr{F}(f)(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ist}f(t)\,dt\ \Big(=\lim_{\alpha,\beta\to\infty}\int_{-\alpha}^{\beta}e^{-ist}f(t)dt\Big)\,,$$

where the kernel K is a complex function (i.e., the value of K is complex). We will discuss the Fourier transform later.

Definition 7.4 (Laplace transform)**.** Let $f:[0,\infty]\to\mathbb{R}$ be a function. The ***Laplace transform*** of f, denoted by $\mathscr{L}(f)$, is defined by

$$\mathscr{L}(f)(s)=\int_{0}^{\infty}e^{-st}f(t)\,dt\ \Big(=\lim_{R\to\infty}\int_{0}^{R}e^{-ist}f(t)dt\Big)\,,$$

provided that the improper integral exists.

Example 7.5. Let $f:[0,\infty)\to\mathbb{R}$ be defined by $f(t)=e^{at}$, where $a\in\mathbb{R}$ is a constant. Since the improper integral

$$\int_0^{\infty}e^{(a-s)t}\,dt=\lim_{R\to\infty}\int_0^R e^{(a-s)t}\,dt\overset{(s\neq a)}{=}\lim_{R\to\infty}\Big(-\frac{e^{(a-s)t}}{(s-a)}\Big|_{t=0}^{t=R}\Big)=\lim_{R\to\infty}\frac{1-e^{(a-s)R}}{s-a}$$

exists for $s>a$, we find that

$$\mathscr{L}(f)(s)=\frac{1}{s-a}\qquad\forall\,s>a\,.$$

Example 7.6. Let $f:[0,\infty)\to\mathbb{R}$ be defined by

$$f(t)=\begin{cases}1 & \text{if } 0\leqslant t<1\,,\\ k & \text{if } t=1\,,\\ 0 & \text{if } t>1\,,\end{cases}$$

where k is a given constant. Since the improper integral

$$\int_0^{\infty}e^{-st}f(t)\,dt=\int_0^1 e^{-st}dt=\frac{1-e^{-s}}{s}$$

exists as long as $s\neq 0$, we find that

$$\mathscr{L}(f)(s)=\frac{1-e^{-s}}{s}\qquad\forall\,s\neq 0\,.$$

We note that the Laplace transform in this case is independent of the choice of k; thus the Laplace transform is not one-to-one (in the classical/pointwise sense).

Example 7.7. Let $f:[0,\infty)\to\mathbb{R}$ be given by $f(t)=\sin(at)$. Note that

$$\begin{aligned}\int_0^R \underbrace{e^{-st}}_{\equiv u}\underbrace{\sin(at)\,dt}_{\equiv dv}&=-e^{-st}\frac{\cos(at)}{a}\Big|_{t=0}^{t=R}+\int_0^R(-s)e^{-st}\frac{\cos(at)}{a}\,dt\\ &=\frac{1}{a}\Big(1-e^{-Rs}\cos(aR)\Big)-\frac{s}{a}\int_0^R e^{-st}\cos(at)\,dt \qquad (7.2)\\ &=\frac{1}{a}\Big(1-e^{-Rs}\cos(aR)\Big)-\frac{s}{a}\Big(e^{-st}\frac{\sin(at)}{a}\Big|_{t=0}^{t=R}+\frac{s}{a}\int_0^R e^{-st}\sin(at)\,dt\Big)\\ &=\frac{1}{a}\Big(1-e^{-Rs}\cos(aR)\Big)-\frac{s}{a^2}e^{-Rs}\sin(aR)-\frac{s^2}{a^2}\int_0^R e^{-st}\sin(at)\,dt\,;\end{aligned}$$

thus we obtain that

$$\Big(1+\frac{s^2}{a^2}\Big)\int_0^R e^{-st}\sin(at)\,dt = \frac{1}{a}\Big(1-e^{-Rs}\cos(aR)\Big) - \frac{s}{a^2}e^{-Rs}\sin(aR)\,.$$

Therefore, the improper integral

$$\begin{aligned}\int_0^\infty e^{-st}\sin(at)\,dt &= \lim_{R\to\infty}\int_0^R e^{-st}\sin(at)\,dt \\ &= \lim_{R\to\infty}\Big[\frac{a}{s^2+a^2}\Big(1-e^{-Rs}\cos(aR)\Big) - \frac{s}{s^2+a^2}e^{-Rs}\sin(aR)\Big]\end{aligned}$$

exists for all $s>0$ which implies that

$$\mathscr{L}(f)(s) = \frac{a}{s^2+a^2} \qquad \forall\, s>0\,.$$

Moreover, (7.2) further implies that

$$\int_0^\infty e^{-st}\cos(at)\,dt = \frac{a}{s}\Big(\frac{1}{a}-\frac{a}{s^2+a^2}\Big) = \frac{s}{s^2+a^2}\,.$$

Proposition 7.8. *Suppose that*

1. *f is piecewise continuous on the interval $0 \leqslant t \leqslant R$ for all positive $R \in \mathbb{R}$;*

2. *f is of exponential order a; that is, $|f(t)| \leqslant Me^{at}$ for some M and a.*

Then the Laplace transform of f exists for $s>a$.

Proof. Since f is piecewise continuous on $[0,R]$, the integral $\int_0^R e^{-st}f(t)\,dt$ exists. If $0<R_1<R_2$, by the fact that $|f(t)| \leqslant Me^{at}$ for some M and a, we find that

$$\Big|\int_{R_1}^{R_2} e^{-st}f(t)\,dt\Big| \leqslant \int_{R_1}^{R_2} e^{-st}Me^{at}\,dt = M\frac{e^{(a-s)R_2}-e^{(a-s)R_1}}{a-s}$$

which converges to 0 as $R_1, R_2 \to \infty$ if $s>a$. Therefore, the improper integral $\int_0^\infty e^{-st}f(t)\,dt$ exists.
□

Example 7.9. Let $f:[0,\infty)\to\mathbb{R}$ be given by $f(t)=t^p$ for some $p>-1$. Recall that the Gamma function $\Gamma:(0,\infty)\to\mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt\,.$$

We note that if $-1<p<0$, f is not of exponential order a for all $a\in\mathbb{R}$; however, the Laplace transform of f exists the Laplace transform of f exists. In fact, for $s>0$,

$$\mathscr{L}(f)(s) = \lim_{R\to\infty}\int_0^R e^{-st}t^p\,dt = \lim_{R\to\infty}\int_0^{sR} e^{-t}\Big(\frac{t}{s}\Big)^p\frac{dt}{s} = \frac{\Gamma(p+1)}{s^{p+1}}\,.$$

In particular, if $p=n\in\mathbb{N}\cup\{0\}$, then

$$\mathscr{L}(f)(s) = \frac{n!}{s^{n+1}} \qquad \forall\, s>0\,.$$

7.1.1 The Inverse Laplace Transform

Even though Example 7.6 shows that the Laplace transform is not one-to-one in the classical sense, we are still able to talk about the "inverse" of the Laplace transform because of the following

Theorem 7.10 (Lerch). *Suppose that $f, g : [0,\infty) \to \mathbb{R}$ are continuous and of exponential order a. If $\mathscr{L}(f)(s) = \mathscr{L}(g)(s)$ for all $s > a$, then $f(t) = g(t)$ for all $t \geqslant 0$.*

Remark 7.11. The ***inverse Laplace transform*** of a function F is given by

$$\mathscr{L}^{-1}(F)(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma-iR}^{\gamma+iR} e^{st} F(s)\, ds\,,$$

where the integration is done along the vertical line $\mathrm{Re}(s) = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of F.

7.2 Solution of Initial Value Problems

Theorem 7.12. *Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous with piecewise continuous derivative, and f is of exponential order a. Then the Laplace transform of f' exist for $s > a$, and*

$$\mathscr{L}(f')(s) = s\mathscr{L}(f)(s) - f(0)\,.$$

Proof. Since f is of exponential order, the Laplace transform of f exists. Since f is continuous, integrating by parts we find that

$$\int_0^R e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_{t=0}^{t=R} - \int_0^R (-s)e^{-st} f(t)\, dt = e^{-Rs} f(R) - f(0) s \int_0^R e^{-st} f(t)\, dt\,.$$

Since f is of exponential order a, $e^{-Rs} f(R) \to 0$ as $s \to \infty$; thus

$$\mathscr{L}(f')(s) = \lim_{R\to\infty} \int_0^R e^{-st} f'(t)\, dt = -f(0) + s \lim_{R\to\infty} \int_0^R e^{-st} f(t)\, dt = s\mathscr{L}(f)(s) - f(0)\,.$$ □

Corollary 7.13. *Suppose that $f : [0,\infty) \to \mathbb{R}$ is a function such that $f, f', f'', \cdots, f^{(n-1)}$ are continuous of exponential order a, and $f^{(n)}$ is piecewise continuous. Then $\mathscr{L}(f^{(n)}(s)$ exists for all $s > a$, and*

$$\mathscr{L}(f^{(n)})(s) = s^n \mathscr{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

Example 7.14. Consider the ODE

$$y'' - y' - 2y = 0\,.$$

If the solution y and its derivative y' are of exponential order a for some $a \in \mathbb{R}$, then by taking the Laplace transform of the equation above we find that

$$\big[s^2 \mathscr{L}(y) - s y(0) - y'(0)\big] - \big[s\mathscr{L}(y) - y(0)\big] - 2\mathscr{L}(y) = 0\,;$$

thus

$$\begin{aligned}\mathscr{L}(y)(s) &= \frac{sy(0)+y'(0)-y(0)}{s^2-s-2} = \frac{sy(0)+y'(0)-y(0)}{(s-2)(s+1)}\\ &= \frac{y(0)}{s+1} + \frac{y'(0)+y(0)}{(s-2)(s+1)} = \frac{y(0)}{s+1} + \frac{y'(0)+y(0)}{3}\Big(\frac{1}{s-2} - \frac{1}{s+1}\Big).\end{aligned}$$

By Example 7.5 and Theorem 7.10, we find that

$$y(t) = y(0)e^{-t} + \frac{y'(0)+y(0)}{3}\big(e^{2t} - e^{-t}\big).$$

The procedure listed in Example 7.14 provides a way of solving of an ODE with constant coefficients. In fact, suppose that we are looking for solutions to

$$y'' + by' + cy = f(t).$$

Then taking the Laplace transform of the equation above (here we assume that y and y' are of exponential order a for some $a \in \mathbb{R}$), we find that

$$s^2\mathscr{L}(y)(s) - sy(0) - y'(0) + b\big(s\mathscr{L}(y)(s) - y(0)\big) + c\mathscr{L}(y)(s) = \mathscr{L}(f)(s)$$

which implies that the Laplace transform of the solution y satisfies

$$\mathscr{L}(y)(s) = \frac{(s+b)y(0)+y'(0)}{s^2+bs+c} + \frac{\mathscr{L}(f)(s)}{s^2+bs+c}. \tag{7.3}$$

The ODE is then solved provided that we can find the function $y = \varphi(t)$ whose Laplace transform is the right-hand side of (7.3).

Example 7.15. Find the solution of the ODE $y'' + y = \sin 2t$ with initial condition $y(0) = 2$ and $y'(0) = 1$. If y is the solution to the ODE and y, y' are of exponential order a for some $a \in \mathbb{R}$, then (7.3) and Example 7.7 imply that the Laplace transform of y is given by

$$\mathscr{L}(y)(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}.$$

Using partial fractions, we expect that

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} = \frac{(a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)}{(s^2+1)(s^2+4)}.$$

Therefore, $a+c = b+d = 4a+c = 0$ and $4b+d = 2$; thus $a = c = 0$ and $b = -d = \frac{2}{3}$. This provides that

$$\mathscr{L}(y)(s) = \frac{2s+1}{s^2+1} + \frac{2}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4} = \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}.$$

By Proposition 7.2 and Example 7.7, we find that

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$

Example 7.16. Find the solution of the ODE $y^{(4)} - y = 0$ with initial condition $y(0) = y''(0) = y'''(0) = 0$ and $y'(0) = 1$ and y, y' are of exponential order a for some $a \in \mathbb{R}$. If y is the solution to the ODE, then Corollary 7.13 implies that the Laplace transform of y satisfies

$$s^4 \mathscr{L}(y)(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - \mathscr{L}(y)(s) = 0\,;$$

thus

$$\mathscr{L}(y)(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s-1)(s+1)(s^2+1)}\,.$$

Using partial fractions, we assume that

$$\begin{aligned}\mathscr{L}(y)(s) = \frac{s^2}{s^4-1} &= \frac{a}{s-1} + \frac{b}{s+1} + \frac{cs+d}{s^2+1} = \frac{(a+b)s + (a-b)}{s^2-1} + \frac{cs+d}{s^2+1} \\ &= \frac{(a+b+c)s^3 + (a-b+d)s^2 + (a+b-c)s + (a-b-d)}{s^4-1}\,.\end{aligned}$$

Therefore, $a + b + c = a + b - c = a - b - d = 0$ and $a - b + d = 1$; thus $a = \frac{1}{4}$, $b = -\frac{1}{4}$, $c = 0$ and $d = \frac{1}{2}$. This provides that

$$\mathscr{L}(y)(s) = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s^2+1}\,.$$

By Example 7.5 and 7.7, we conclude that the solution to the ODE is

$$y(t) = \frac{1}{4}e^t - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t\,.$$

- **Advantages of the Laplace transform method**:

1. Converting a problem of solving a differential equation to a problem of solving an algebraic equation.
2. The dependence on the initial data is automatically build in. The task of determining values of arbitrary constants in the general solution is avoided.
3. Non-homogeneous equations can be treated in exactly the same way as the homogeneous ones, and it is not necessary to solving the corresponding homogeneous equation first.

- **Difficulties of the Laplace transform method**: Need to find the function whose Laplace transform is given - the inverse Laplace transform has to be performed in general situations.

7.3 Step Functions

In the following two sections we are concerned with the Laplace transform of discontinuous functions with jump discontinuities.

Definition 7.17. The ***unit step function*** or ***Heaviside function*** is the function

$$H(t) = \begin{cases} 0 & \text{if } t < 0\,, \\ 1 & \text{if } t \geqslant 0\,. \end{cases}$$

Example 7.18.

1. For $c \in \mathbb{R}$, we define $u_c(t) = H(t-c)$. Then the graph of u_c jumps up from 0 to 1 at $t = c$.

2. The graph of $-u_c$ jumps down from 1 to 0 at $t = c$.

3. Let $a < b$. The characteristic/indicator function $\mathbf{1}_{[a,b)}$ can be expressed by

$$\mathbf{1}_{[a,b)}(t) = u_a(t) - u_b(t)\,.$$

4. Let $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$. The step function

$$f(t) = \sum_{i=1}^{n} f_i \mathbf{1}_{[a_i,b_i)}(t) \tag{7.4}$$

can be expressed by

$$f(t) = \sum_{i=1}^{n} f_i\big[u_{a_i}(t) - u_{b_i}(t)\big]\,.$$

5. Let $0 = c_0 < c_1 < \cdots < c_n < c_{n+1} = \infty$. The step function

$$f(t) = \sum_{i=0}^{n} f_i \mathbf{1}_{[c_i,c_{i+1})}(t) \tag{7.5}$$

can be expressed by

$$f(t) = f_0 \mathbf{1}_{[0,c_1)}(t) + \sum_{k=0}^{n} (f_{k+1} - f_k) u_{c_k}(t)\,.$$

• **The Laplace transform of** u_c: To compute the Laplace transform of the step function given by (7.4), by Proposition 7.2 it suffices to find the Laplace transform of u_c.

1. If $c \leqslant 0$, then

$$\mathscr{L}(u_c)(s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s} \qquad \forall\, s > 0\,.$$

2. If $c > 0$, then

$$\mathscr{L}(u_c)(s) = \int_c^\infty e^{-st}\,dt = \frac{e^{-cs}}{s} \qquad \forall\, s > 0\,.$$

Therefore,

$$\mathscr{L}(u_c)(s) = \frac{e^{-\max\{c,0\}s}}{s}\,.$$

Theorem 7.19. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that the Laplace transform $\mathscr{L}(f)(s)$ of f exists for $s > a \geqslant 0$. If c is a positive constant and $g(t) = u_c(t)f(t-c)$, then*

$$\mathscr{L}(g)(s) = e^{-cs}\mathscr{L}(f)(s)\,.$$

Conversely, if $G(s) = e^{-cs}\mathscr{L}(f)(s)$, then $u_c(t)f(t-c) = \mathscr{L}^{-1}(G)(t)$.

Proof. If $c > 0$ and $g(t) = u_c(t)f(t-c)$, then the change of variable formula implies that

$$\begin{aligned}\mathscr{L}(g)(s) &= \lim_{R\to\infty}\int_c^R e^{-st}f(t-c)\,dt = \lim_{R\to\infty}\int_0^{R-c} e^{-s(t+c)}f(t)\,dt \\ &= e^{-cs}\lim_{R\to\infty}\int_0^{R-c} e^{-st}f(t)\,dt = e^{-cs}\mathscr{L}(f)(s)\,. \end{aligned}$$ □

Example 7.20. Let $f : [0,\infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} \sin t & \text{if } 0 \leqslant t < \dfrac{\pi}{4}\,, \\ \sin t + \cos\big(t - \dfrac{\pi}{4}\big) & \text{if } t \geqslant \dfrac{\pi}{4}\,. \end{cases}$$

Then $f(t) = \sin t + u_{\frac{\pi}{4}}(t)\cos\big(t - \frac{\pi}{4}\big)$; thus by Example 7.7 and Theorem 7.19 we find that

$$\mathscr{L}(f)(s) = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\frac{s}{s^2+1} = \frac{1+se^{-\frac{\pi}{4}s}}{s^2+1}\,.$$

Example 7.21. Find the inverse Laplace transform of $F(s) = \dfrac{1-e^{-2s}}{s^2}$.

By Example 7.9, the inverse Laplace transform of s^{-2} is $\dfrac{t}{\Gamma(1+1)} = t$; thus Theorem 7.19 implies that

$$\mathscr{L}^{-1}(F)(t) = t - u_2(t)(t-2)\,.$$

We also have the following

Theorem 7.22. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that the Laplace transform $\mathscr{L}(f)(s)$ of f exists for $s > a \geqslant 0$. If c is a constant and $g(t) = e^{ct}f(t)$, then*

$$\mathscr{L}(g)(s) = \mathscr{L}(f)(s-c) \qquad \forall\, s > a + c\,.$$

Conversely, if $G(s) = \mathscr{L}(f)(s-c)$, then $\mathscr{L}^{-1}(G)(t) = e^{ct}f(t)$.

Proof. By the definition of the Laplace transform,

$$\mathscr{L}(g)(s) = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = \mathscr{L}(f)(s-c)\,.$$ □

Example 7.23. Find the inverse Laplace transform of $G(s) = \dfrac{1}{s^2-4s+5}$.

By completing the square, $s^2 - 4s + 5 = (s-2)^2 + 1$; thus Example 7.7 and Theorem 7.22 implies that

$$\mathscr{L}^{-1}(G)(t) = e^{2t}\sin t\,.$$

7.4 Differential Equations with Discontinuous Forcing Functions

Let $f : [0,\infty) \to \mathbb{R}$ be a function defined by

$$f(t) = \begin{cases} f_1(t) & \text{if } 0 \leqslant t < c\,, \\ f_2(t) & \text{if } t \geqslant c\,, \end{cases}$$

where f_1, f_2 are continuous and $\lim_{t\to c^+} f_2(t) - \lim_{t\to c^-} f_1(t) = A$ (such a point c is called a jump discontinuity of f). Define

$$g(t) = \begin{cases} f_1(t) & \text{if } 0 \leqslant t < c\,, \\ f_2(t) - Au_c(t) & \text{if } t \geqslant c\,. \end{cases}$$

Then $g : [0, \infty) \to \mathbb{R}$ is continuous, and $f = g + Au_c$. Similarly, if f is a piecewise continuous function which only has jump discontinuities $\{c_1, c_2, \cdots, c_n\}$ such that f is continuous on $[c_k, c_{k+1})$ for all $k \in \{1, \cdots, n-1\}$. Then By introducing $c_0 = 0$ and $c_{n+1} = \infty$, we can write

$$f = f\mathbf{1}_{[c_0,c_1)} + f\mathbf{1}_{[c_1,c_2)} + \cdots + f\mathbf{1}_{[c_{n-1},c_n)} + f\mathbf{1}_{[c_n,c_{n+1})}\,.$$

If $A_k \equiv \lim_{t\to c_k^+} (f\mathbf{1}_{[c_k,c_{k+1})})(t) - \lim_{t\to c_k^-} (f\mathbf{1}_{[c_{k-1},c_k)})(t)$, then the function $g : [0, \infty) \to \mathbb{F}$ defined by

$$g(t) = f(t) - \sum_{k=1}^{n} A_k u_{c_k}(t)$$

is continuous on $\mathbb{R}$, and $f = g + \sum\limits_{k=1}^{n} A_k u_{c_k}$.

Now suppose that we are looking for a solution to

$$y'' + by' + cy = f(t)\,, \tag{7.6}$$

where f is a piecewise continuous function which only has jump discontinuities $\{c_1, c_2, \cdots, c_n\}$ as described above. We note that the existence theorem (Theorem 2.10) cannot be applied due to the discontinuity of the forcing function, so in general we do not know if a solution exists. However, if there indeed exists a twice differentiable function y validating (7.6), then the solution must be unique since if y_1 and y_2 are two solutions with the same initial condition, then $y = y_1 - y_2$ is a solution to $y'' + by' + cy = 0$ with $y(0) = y'(0) = 0$; thus y must be zero which implies that the solution, if it exists, must be unique. On the other hand, if (7.6) has a solution y, then y'' must be piecewise continuous. If in addition y and y' are of exponential order a for some $a \in \mathbb{R}$, we can apply Theorem 7.13 to find the Laplace transform of the solution y as introduced in Section 7.2 which in principle provides information of how the solution can be found.

Now we focus on solving the ODE

$$y'' + by' + cy = F\mathbf{1}_{[\alpha,\beta)}(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \tag{7.7}$$

where F is a constant and $0 < \alpha < \beta$. We only consider the case that $c \neq 0$ for otherwise the ODE can reduced to a first order ODE (by integrating the ODE). We note that the right-hand side can also be written as $F\big[u_\alpha(t) - u_\beta(t)\big]$.

If y is a twice differentiable solution to (7.7), taking the Laplace transform of the ODE we find that

$$s^2\mathscr{L}(y)(s) - sy_0 - y_1 + b\big[s\mathscr{L}(y)(s) - y_0\big] + c\mathscr{L}(y)(s) = F\frac{e^{-\alpha s} - e^{-\beta s}}{s}\,;$$

thus

$$\mathscr{L}(y)(s) = \frac{(s+b)y_0 + y_1}{s^2 + bs + c} + F\frac{e^{-\alpha s} - e^{-\beta s}}{s(s^2 + bs + c)}\,.$$

Using partial fractions, we obtain that $\dfrac{1}{s(s^2+bs+c)} = \dfrac{1}{c}\Big[\dfrac{1}{s} - \dfrac{s+b}{s^2+bs+c}\Big]$; thus with z denoting the solution to the ODE

$$z'' + bz' + cz = 0\,, \qquad z(0) = 1\,, \quad z'(0) = 0\,,$$

we find that

$$\frac{e^{-\alpha s} - e^{-\beta s}}{s(s^2+bs+c)} = \frac{e^{-\alpha s} - e^{-\beta s}}{c}\mathscr{L}(1-z)(s)\,.$$

Therefore, Theorem 7.19 implies that

$$y(t) = Y(t) + \frac{F}{c}\Big[\big[u_\alpha(t)(1-z(t-\alpha)\big] - u_\beta(t)\big[1-z(t-\beta)\big]\Big]\,, \tag{7.8}$$

here Y is the solution to (7.7) with $F = 0$. The function y given in (7.8) is the only possible solution to (7.7). We note that even though u_α, $\boldsymbol{u}_\beta$ are discontinuous at $t = \alpha, \beta$, the function y given in (7.8) is continuous for all t since $z(0) = 1$.

- **The first derivative of** y: For $t \neq \alpha, \beta$, it is clear that $y'(t)$ exists and can be computed by

$$y'(t) = Y'(t) + \frac{F}{c}\big[u_\beta(t)z'(t-\beta) - u_\alpha(t)z'(t-\alpha)\big]\,. \tag{7.9}$$

Now we check the differentiability of y at $t = \alpha$ and $t = \beta$ by looking at the limits

$$\lim_{h\to 0^-}\frac{y(c+h)-y(c)}{h} \quad \text{and} \quad \lim_{h\to 0^+}\frac{y(c+h)-y(c)}{h} \qquad \text{for} \quad c = \alpha, \beta\,.$$

For $|h| \ll 1$, $\alpha + h < \beta$. Therefore, by the differentiability of Y,

$$\lim_{h\to 0^-}\frac{y(\alpha+h)-y(\alpha)}{h} = Y'(\alpha) + \frac{F}{c}\lim_{h\to 0^-}\frac{u_\alpha(\alpha+h)\big(1-z(h)\big) - u_\alpha(\alpha)\big(1-z(0)\big)}{h} = Y'(\alpha)$$

and

$$\begin{aligned}
\lim_{h\to 0^+}\frac{y(\alpha+h)-y(\alpha)}{h} &= Y'(\alpha) + \frac{F}{c}\lim_{h\to 0^+}\frac{u_\alpha(\alpha+h)\big(1-z(h)\big) - u_\alpha(\alpha)\big(1-z(0)\big)}{h}\\
&= Y'(\alpha) + \frac{F}{c}\lim_{h\to 0^+}\frac{1-z(h)}{h} = Y'(\alpha) - \frac{F}{c}\frac{z(h)-z(0)}{h}\\
&= Y'(\alpha) - \frac{F}{c}z'(0) = Y'(\alpha)\,.
\end{aligned}$$

Therefore, y' exists at $t = \alpha$ and $y'(\alpha) = Y'(\alpha)$ which also validates (7.9) for $t = \alpha$. Similarly,

$$y'(\beta) = Y'(\beta) - \frac{F}{c}\big[u_\beta'(\alpha)(1-z(\beta-\alpha)\big) + u_\alpha(\beta)z'(\beta-\alpha)\big] = Y'(\beta) - \frac{F}{c}z'(\beta-\alpha)$$

since

$$\lim_{h\to 0}\frac{u_\beta(\beta+h)\big(1-z(h)\big) - u_\beta(\beta)\big(1-z(0)\big)}{h} = 0\,.$$

In other words, (7.9) holds for all $t > 0$. We note that y' given by (7.9) is continuous since

$$\lim_{y\to\alpha} y'(t) = Y'(\alpha) = y'(\alpha)$$

and

$$\lim_{y\to\beta} y'(t) = Y'(\beta) - \frac{F}{c} z'(\beta-\alpha) + \frac{F}{c}\lim_{y\to\beta} u_\beta(t) z'(t-\beta) = Y'(\beta) - \frac{F}{c} z'(\beta-\alpha) = y'(\beta)\,.$$

• **The second derivative of** y: Now we turn our attention to the second derivative of y. As before, it suffices to check the differentiability of y' at $t=\alpha,\beta$ since

$$y''(t) = Y''(t) + \frac{F}{c}\big[u_\beta(t) z''(t-\beta) - u_\alpha(t) z''(t-\alpha)\big] \qquad \forall\, t>0\,, t\neq \alpha,\beta\,. \tag{7.10}$$

For $t=\alpha$, we find that

$$\lim_{h\to 0^-} \frac{y'(\alpha+h)-y'(\alpha)}{h} = Y''(\alpha) - \frac{F}{c}\lim_{h\to 0^-}\frac{u_\alpha(\alpha+h)z'(h) - u_\alpha(\alpha)z'(0)}{h} = Y''(\alpha)$$

and

$$\begin{aligned}
\lim_{h\to 0^+} \frac{y'(\alpha+h)-y'(\alpha)}{h} &= Y''(\alpha) - \frac{F}{c}\lim_{h\to 0^+}\frac{u_\alpha(\alpha+h)z'(h) - u_\alpha(\alpha)z'(0)}{h}\\
&= Y''(\alpha) - \frac{F}{c}\lim_{h\to 0^+}\frac{z'(h)-z'(0)}{h} = Y''(\alpha) - \frac{F}{c} z''(0)\\
&= Y''(\alpha) + \frac{F}{c}\big[bz'(0)+cz(0)\big] = Y''(\alpha) + F\,.
\end{aligned}$$

Since $F\neq 0$, we conclude that the second derivative of y at $t=\alpha$ does not exist. Similarly, the second derivative of y at $t=\beta$ does not exist neither. Nevertheless, for $t\neq\alpha,\beta$,

$$\begin{aligned}
y''(t) + by'(t) + cy(t) = Y''(t) &+ \frac{F}{c}\big[u_\beta(t) z''(t-\beta) - u_\alpha(t) z''(t-\alpha)\big]\\
&+ bY'(t) + \frac{bF}{c}\big[u_\beta(t) z'(t-\beta) - u_\alpha(t) z'(t-\alpha)\big]\\
&+ cY(t) + F\Big[\big[u_\alpha(t)(1-z(t-\alpha)\big] - u_\beta(t)\big[1-z(t-\beta)\big]\Big]\\
&= \frac{F}{c}\Big[u_\alpha(t)\big[bz'(t-\alpha)+cz(t-\alpha)\big] - u_\beta(t)\big[bz'(t-\beta)+cz(t-\beta)\big]\Big]\\
&+ \frac{F}{c}\big[bu_\beta(t) z'(t-\beta) - bu_\alpha(t) z'(t-\alpha)\big]\\
&+ \frac{F}{c}\Big[cu_\alpha(t)\big[(1-z(t-\alpha)\big] - cu_\beta(t)\big[1-z(t-\beta)\big]\Big]\\
&= F\big[u_\alpha(t) - u_\beta(t)\big] = F\mathbf{1}_{[\alpha,\beta)}(t)\,.
\end{aligned}$$

Summary: There is no function which validates (7.7) for all $t>0$. However, there exists a continuously differentiable function whose second derivative is piecewise continuous which validates (7.7) for all $t>0$ except the discontinuities of the second derivative. We shall also call such a function a solution to (7.7).

Definition 7.24. Let $f:[0,\infty)\to\mathbb{R}$ be a function. A function y is said to be a solution to the ODE

$$y''+by'+cy = f(t) \qquad y(0)=y_0,\quad y'(0)=y_1$$

if y is continuously differentiable, and y'' exists at every continuity of f.

Example 7.25. Find the solution of the ODE $y'' + 4y = g(t)$ with initial data $y(0) = y'(0) = 0$, where the forcing function g is given by

$$g(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < 5\,, \\ \dfrac{t-5}{5} & \text{if } 5 \leqslant t < 10\,, \\ 1 & \text{if } t \geqslant 10\,. \end{cases}$$

We note that $g(t) = \dfrac{1}{5}\big[u_5(t)(t-5) - u_{10}(t)(t-10)\big]$; thus Example 7.9 and Theorem 7.19 imply that

$$\mathscr{L}(g)(s) = \frac{1}{5}\frac{1}{s^2}(e^{-5s} - e^{-10s}) = \frac{e^{-5t} - e^{-10t}}{5s^2}\,.$$

We also remark that $g'(t) = \dfrac{1}{5}(u_5(t) - u_{10}(t))$ if $t \neq 5, 10$. Since the value at g' at two points does not affect the Laplace transform, we can use Corollary 7.13 to compute the Laplace transform of g:

$$s\mathscr{L}(g)(s) = s\mathscr{L}(g)(s) - g(0) = \mathscr{L}(g')(s) = \frac{e^{-5t} - e^{-10t}}{5s}\,;$$

thus $\mathscr{L}(g)(s) = \dfrac{e^{-5s} - e^{-10s}}{5s^2}$.

Assume that a solution y to the ODE under consideration exists such that y, y' are continuous and y'' are of exponential order a for some $a \in \mathbb{R}$. Then the Laplace transform implies that

$$s^2\mathscr{L}(y)(s) - sy(0) - y'(0) + 4\mathscr{L}(y)(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}\,.$$

Therefore,

$$\mathscr{L}(y)(s) = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)}\,.$$

Using partial fractions, we assume that $\dfrac{1}{s^2(s^2+4)} = \dfrac{as+b}{s^2} + \dfrac{cs+d}{s^2+4}$, where a, b, c, d satisfy $a + c = 0$, $b + d = 0$, $4a = 0$ and $4b = 1$; thus

$$\mathscr{L}(y)(s) = \frac{e^{-5s} - e^{-10s}}{20}\Big[\frac{1}{s^2} - \frac{1}{2}\frac{2}{s^2+4}\Big]\,.$$

By Theorem 7.22, we find that

$$y(t) = \frac{1}{20}\Big[u_5(t)(t-5) - u_{10}(t)(t-10) - \frac{1}{2}\Big(u_5(t)\sin\big(2(t-5)\big) - u_{10}(t)\sin\big(2(t-10)\big)\Big)\Big]\,.$$

Remark 7.26. The Laplace transform picks up solutions whose derivative of the highest order (which is the same as the order of the ODE under consideration) is of exponential order a for some $a \in \mathbb{R}$.

7.5 Impulse Functions

In this section, we are interested in what happens if a moving object in a spring-mass system is hit by an external force which only appears in a very short amount of time period (you can think of hitting an object in a spring-mass system using a hammer in a very short amount of time). In practice, we do not know the exact time period $[\alpha, \beta]$ (with $|\beta - \alpha| \ll 1$) during which the force hits the system, but can assume that the total amount of force which affects the system is known. This kind of phenomena usually can be described by the system

$$y'' + by' + cy = f(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1$$

for some special kind of functions f which has the following properties:

1. f is sign-definite; that is, $f(t) \geqslant 0$ for all $t > 0$ or $f(t) \leqslant 0$ for all $t < 0$;

2. f is and is supported in $[t_0 - \tau, t_0 + \tau]$ for some $t_0 > 0$ and some very small $\tau > 0$;

3. $\displaystyle\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt = F$, where F is a constant independent of τ.

This kind of force is called an ***impulse***.

Example 7.27. Let $d_\tau : \mathbb{R} \to \mathbb{R}$ be a step function defined by

$$d_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & \text{if } t \in [-\tau, \tau)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{7.11}$$

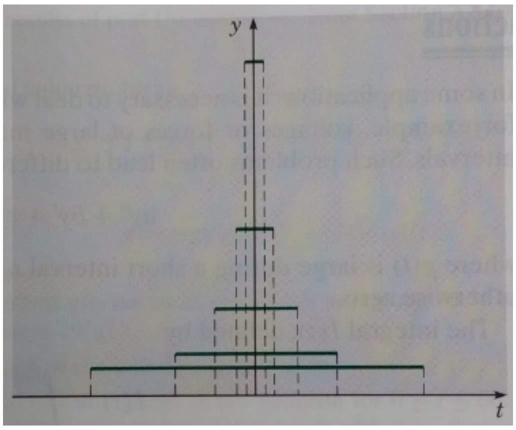


Figure 4: The graph of $y = d_\tau(t)$ as $\tau \to 0^+$.

Then $f(t) = F d_\tau(t)$ is an impulse function. We note that with d denoting the function $\dfrac{1}{2}\mathbf{1}_{[-1,1)}$, then $d_\tau(t) = \dfrac{1}{\tau} d\big(\dfrac{t}{\tau}\big)$. Moreover, if $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous in an open interval containing 0, we must have

$$\lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t)\varphi(t)\,dt = \varphi(0)\,. \tag{7.12}$$

Example 7.28. Let

$$\eta(t) = \begin{cases} C \exp\big(\dfrac{1}{t^2 - 1}\big) & \text{if } |t| < 1\,, \\ 0 & \text{if } |t| \geqslant 1\,, \end{cases}$$

where C is chosen so that the integral of η is 1. Then the sequence $\{\eta_\tau\}_{\tau>0}$ defined by

$$\eta_\tau(t) = \frac{1}{\tau}\eta\Big(\frac{t}{\tau}\Big) \tag{7.13}$$

also has the property that

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty}\eta_\tau(t)\varphi(t)\,dt = \varphi(0) \tag{7.14}$$

for all $\varphi : \mathbb{R}\to\mathbb{R}$ which is continuous in an open interval containing 0.

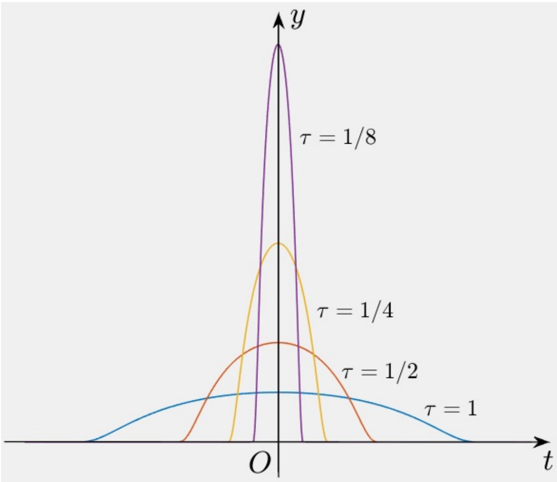


Figure 5: The graph of η_τ for $\tau = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

To see this, we notice that η_τ is supported in $[-\tau,\tau]$ and the integral of η_τ is still 1. Suppose that $\varphi : \mathbb{R}\to\mathbb{R}$ is continuous on (a,b) for some $a<0<b$. Then there exists $0<\delta<\min\{-a,b\}$ such that

$$\big|\varphi(t)-\varphi(0)\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |t|<\delta\,.$$

Therefore, if $0<\tau<\delta$, by the non-negativity of η_τ we find that

$$\begin{aligned}\Big|\int_{-\infty}^{\infty}\eta_\tau(t)\varphi(t)-\varphi(0)\Big| &= \Big|\int_{-\tau}^{\tau}\eta_\tau(t)\varphi(t)\,dt-\varphi(0)\int_{-\tau}^{\tau}\eta_\tau(t)\,dt\Big| \\ &= \int_{-\tau}^{\tau}\eta_\tau(t)\big[\varphi(t)-\varphi(0)\big]\,dt \\ &\leqslant \int_{-\tau}^{\tau}\eta_\tau(t)\big|\varphi(t)-\varphi(0)\big|\,dt \leqslant \frac{\varepsilon}{2}\int_{-\tau}^{\tau}\eta_\tau(t)\,dt < \varepsilon\end{aligned}$$

which validates (7.14).

Definition 7.29. A sequence of functions $\{\zeta_\tau\}_{\tau>0}$, where $\zeta_\tau : \mathbb{R}\to\mathbb{R}$ for all $\tau>0$, is called an ***approximation of the identity*** if $\{\zeta_\tau\}_{\tau>0}$ satisfies

1. $\zeta_\tau(t)\geqslant 0$ for all $t\in\mathbb{R}$.
2. $\lim\limits_{\tau\to 0^+}\int_{-\infty}^{\infty}\zeta_\tau(t)\,dt = 1$.

3. For all $\delta > 0$, $\displaystyle\lim_{\tau \to 0^+} \int_{|t|>\delta} \zeta_\tau(t)\,dt = 0$.

In particular, $\{d_\tau\}_{\tau>0}$ and $\{\eta_\tau\}_{\tau>0}$ are approximations of identity.

Using the same technique of establishing (7.14), one can also prove that if $\{\zeta_\tau\}_{\tau>0}$ is an approximation of the identity, then

$$\lim_{\tau \to 0} \int_{-\infty}^{\infty} \zeta_\tau(t)\varphi(t)\,dt = \varphi(0)\,.$$

Remark 7.30. An approximation of identities does not have to be compactly supported. For example, let $n(t) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ be the probability density function of the normal distribution $N(0,1)$, then $n_\tau(t) \equiv \dfrac{1}{\sqrt{2\pi\tau}} e^{-\frac{t^2}{2\tau}}$ constitutes an approximation of the identity $\{n_\tau\}_{\tau>0}$.

For $t_0 > 0$, we consider the ODE

$$y'' + by' + cy = F d_\tau(t - t_0)\,, \qquad y_\tau(0) = y_0\,, \quad y_\tau'(0) = y_1\,. \tag{7.15}$$

For each $0 < \tau < t_0$, let y_τ be the solution to (7.15). Using (7.8) we find that

$$y_\tau(t) = y_\infty(t) + \frac{F}{2c\tau}\Big[u_{t_0-\tau}(t)\big[1 - z(t - t_0 + \tau)\big] - u_{t_0+\tau}(t)\big[1 - z(t - t_0 - \tau)\big]\Big]\,,$$

where y_∞ is the unique $\mathscr{C}^2$-function solving

$$y_\infty'' + by_\infty' + cy_\infty = 0\,, \qquad y_\infty(0) = y_0\,, \quad y_\infty'(0) = y_1\,.$$

and z is the unique $\mathscr{C}^2$-function solving

$$z'' + bz' + cz = 0\,, \qquad y_\infty(0) = 1\,, \quad y_\infty'(0) = 0\,.$$

We remark here that y_∞, y_∞', z' and z'' are of exponential order a for some $a \in \mathbb{R}$. We also recall that the discussion in Section 7.4 shows that y_τ is continuously differentiable, and y_τ'' is piecewise continuous. Our "goal" here is to find a function y which is independent of τ but $|y - y_\tau| \ll 1$ when $\tau \ll 1$. In other words, our goal is to show that $\{y_\tau\}_{\tau>0}$ converges and find the limit of $\{y_\tau\}_{\tau>0}$.

We claim that $\{y_\tau\}_{\tau>0}$, viewing as functions defined on $[0,T]$, is uniformly bounded and equicontinuous (so that we can extract a uniformly convergent subsequence). To see this, using the identity that

$$u_{a+b}(t) = u_a(t - b) \tag{7.16}$$

we rewrite y_τ as

$$\begin{aligned}
y_\tau(t) &= y_\infty(t) + \frac{F}{2c\tau}\Big[u_{t_0}(t+\tau)\big[1 - z(t - t_0 + \tau)\big] - u_{t_0}(t-\tau)\big[1 - z(t - t_0 - \tau)\big]\Big] \\
&= y_\infty(t) + \frac{F}{2c} \cdot \frac{\big[u_{t_0}(t+\tau) - u_{t_0}(t-\tau)\big]\big[1 - z(t - t_0)\big]}{\tau} \\
&\quad + \frac{F}{2c} \cdot \frac{u_{t_0}(t+\tau)\big[z(t - t_0) - z(t - t_0 + \tau)\big]}{\tau} - \frac{F}{2c} \cdot \frac{u_{t_0}(t-\tau)\big[z(t - t_0) - z(t - t_0 - \tau)\big]}{\tau}\,.
\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}
1 - z(t-t_0) = z(0) - z(t-t_0) = (t_0 - t) z'(\xi_1) &\quad \text{for some } \xi_1 \text{ in between 0 and } t - t_0\,,\\
z(t-t_0) - z(t-t_0+\tau) = z'(\xi_2)\tau &\quad \text{for some } \xi_2 \in (t-t_0, t-t_0+\tau)\,,\\
z(t-t_0) - z(t-t_0-\tau) = z'(\xi_3)\tau &\quad \text{for some } \xi_3 \in (t-t_0-\tau, t-t_0)\,;
\end{aligned}$$

thus

1. The case $-\tau < t_0 - t \leqslant \tau$: in this case we have $u_{t_0}(t+\tau) - u_{t_0}(t-\tau) = 1$; thus

$$\begin{aligned}
|y_\tau(t)| &\leqslant |y_\infty(t)| + \Big|\frac{F}{2c}\Big|\Big[|z'(\xi_1)|\frac{|t-t_0|}{\tau} + |z'(\xi_2)| + |z'(\xi_3)|\Big]\\
&\leqslant |y_\infty(t)| + \Big|\frac{F}{2c}\Big|\Big[|z'(\xi_1)| + |z'(\xi_2)| + |z'(\xi_3)|\Big]\\
&\leqslant \max_{t\in[0,T]} |y_\infty(t)| + \Big|\frac{3F}{2c}\Big| \max_{t\in[0,T]} |z'(t)|\,.
\end{aligned}$$

2. The case $t_0 - t \notin (-\tau, \tau]$: in this case we have $u_{t_0}(t+\tau) - u_{t_0}(t-\tau) = 0$; thus

$$|y_\tau(t)| \leqslant |y_\infty(t)| + \Big|\frac{F}{2c}\Big|\Big[|z'(\xi_2)| + |z'(\xi_3)|\Big] \leqslant \max_{t\in[0,T]} |y_\infty(t)| + \Big|\frac{F}{c}\Big| \max_{t\in[0,T]} |z'(t)|\,. \tag{7.17}$$

Therefore, for all $\tau > 0$ we have

$$\max_{t\in[0,T]} |y_\tau(t)| \leqslant \max_{t\in[0,T]} |y_\infty(t)| + \Big|\frac{3F}{2c}\Big| \max_{t\in[0,T]} |z'(t)| \leqslant C_1 e^{aT}$$

which implies that the sequence $\{y_\tau(t)\}_{\tau>0}$ is uniformly bounded on $[0,T]$ and $\{y_\tau\}_{\tau>0}$ are of exponential order a.

On the other hand, using (7.9) and (7.16) we have

$$\begin{aligned}
y'_\tau(t) &= y'_\infty(t) + \frac{F}{2c\tau}\big[u_{t_0}(t-\tau)z'(t-t_0-\tau) - u_{t_0}(t+\tau)z'(t-t_0+\tau)\big]\\
&= y'_\infty(t) - \frac{F}{2c}\cdot\frac{\big[u_{t_0}(t+\tau) - u_{t_0}(t-\tau)\big]z'(t-t_0)}{\tau}\\
&\quad + \frac{F}{2c}\cdot\frac{u_{t_0}(t-\tau)\big[z'(t-t_0-\tau) - z'(t-t_0)\big]}{\tau}\\
&\quad - \frac{F}{2c}\cdot\frac{u_{t_0}(t+\tau)\big[z'(t-t_0+\tau) - z'(t-t_0)\big]}{\tau}\,.
\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}
z'(t-t_0) = z'(t-t_0) - z'(0) = z''(\eta_1)(t-t_0) &\quad \text{for some } \eta_1 \text{ in between 0 and } t-t_0\,,\\
z'(t-t_0-\tau) - z'(t-t_0) = -z'(\eta_2)\tau &\quad \text{for some } \eta_2 \text{ in } (t-t_0-\tau, t-t_0)\,,\\
z'(t-t_0+\tau) - z'(t-t_0) = z'(\eta_3)\tau &\quad \text{for some } \eta_3 \text{ in } (t-t_0, t-t_0+\tau)\,,
\end{aligned}$$

where we use $z'(0) = 0$ to conclude the existence of η_1. Similar argument used to conclude that $\{y_\tau\}_{\tau>0}$ is uniformly bounded can then be applied to conclude that

$$\max_{t\in[0,T]} \big|y'_\tau(t)\big| \leqslant \max_{t\in[0,T]} \big|y'_\infty(t)\big| + \Big|\frac{3F}{2c}\Big| \max_{t\in[0,T]} |z''(t)| \leqslant C_2 e^{aT}\,.$$

This implies that $\{y_\tau\}_{\tau>0}$ is uniformly Lipschitz and are of exponential order a; thus $\{y_\tau\}_{\tau>0}$, viewed as a sequence of functions defined on $[0,T]$, is equi-continuous. By the Arzelà-Ascoli theorem (Theorem 6.42), there exists a subsequence $\{y_{\tau_j}\}_{j=1}^\infty$ which converges to y uniformly on $[0,T]$ as $j\to\infty$. We note that y is a function defined on $[0,T]$.

Now, by the uniform boundedness and equi-continuity of $\{y_{\tau_j}\}_{j=1}^\infty$ on $[0,T+1]$, there exists a subsequence $\{y_{\tau_{j_\ell}}\}_{\ell=1}^\infty$ which converges to y^* uniformly on $[0,T+1]$. Same procedure provides a further subsequence $\{y_{\tau_{j_{\ell_k}}}\}_{k=1}^\infty$ which converges to y^{**} uniformly on $[0,T+2]$. We note that $y^{**}=y^*$ on $[0,T+1]$ and $y^{**}=y$ on $[0,T]$). We continue this process and obtain a sequence, still denoted by $\{y_{\tau_j}\}_{j=1}^\infty$, and a continuous function $y:[0,\infty)\to\mathbb{R}$ such that $\{y_{\tau_j}\}_{j=1}^\infty$ converges to y uniformly on $[0,T]$ for all $T>0$. We note that (7.17) implies that the limit function y is of exponential order a for some $a>0$. Moreover, we also note that it is still possible that there is another convergent subsequence which converges to another limit function, but we will show that there is only one possible limit function.

Let $\varphi:[0,\infty)\to\mathbb{R}$ be a twice continuously differentiable function which vanishes outside $[0,T]$ for some $T>t_0$. Multiplying the equation above by φ and then integrating on $[0,T]$, we find that

$$\int_0^T (y''_{\tau_j}+by'_{\tau_j}+cy_{\tau_j})\varphi(t)\,dt = F\int_0^T d_{\tau_j}(t-t_0)\varphi(t)\,dt\,.$$

Integrating by parts (twice if necessary) and making a change of variable on the right-hand side,

$$y_0\varphi'(0)-\big(y_1+by_0\big)\varphi(0)+\int_0^\infty y_{\tau_j}(t)\big(\varphi''(t)-b\varphi'(t)+c\varphi(t)\big)\,dt = F\int_{-\infty}^\infty d_{\tau_j}(t)\varphi(t+t_0)\,dt \tag{7.18}$$

for all twice continuously differentiable functions φ vanishing outside some interval $[0,T]$. We note that the integral in (7.18) is not an improper integral but indeed an integral on a bounded interval. Passing to the limit as $j\to\infty$ in (7.18), the uniform convergence of $\{y_{\tau_j}\}_{j=1}^\infty$ to y on any closed interval $[0,T]$ and (7.12) imply that

$$y_0\varphi'(0)-\big(y_1+by_0\big)\varphi(0)+\int_0^\infty y(t)\big(\varphi''(t)-b\varphi'(t)+c\varphi(t)\big)\,dt = F\varphi(t_0) \tag{7.19}$$

for all twice continuously differentiable functions φ vanishing outside some interval $[0,T]$. Since y is of exponential order a, (7.19) in fact holds for all twice continuously differentiable function φ which approaches 0 fast enough at infinity, here the sentence "φ approaches 0 fast enough at infinity" means that

$$\big|\varphi(t)\big|+\big|\varphi'(t)\big|+\big|\varphi''(t)\big|\leqslant Me^{-\tilde{a}t}\qquad\forall\, t\gg 1$$

for some $\tilde{a}>a$.

To see what a possible limit function y is, we let $\varphi(t)=e^{-st}$ for $s>a$ in (7.19) and obtain that

$$-sy_0-(y_1+by_0)+(s^2+bs+c)\int_0^\infty y(t)e^{-st}\,dt = Fe^{-st_0}$$

which, by the definition of the Laplace transform, implies that

$$(s^2+bs+c)\mathscr{L}(y)(s)=(s+b)y_0+y_1+Fe^{-st_0}\,. \tag{7.20}$$

Since every possible limit y of $\{y_\tau\}_{\tau>0}$ is continuous and is of exponential order a, by Theorem 7.10 we conclude that there is only one uniform limit of $\{y_\tau\}_{\tau>0}$; thus $\{y_\tau\}_{\tau>0}$ converges to y uniformly on $[0,T]$ for every $T>0$. By Theorem 7.19 and 7.22, identity (7.20) implies the following:

1. if $r^2+br+c=0$ has two distinct real roots r_1 and r_2, then the solution y to (7.20) is

$$\begin{aligned} y(t) &= y_\infty(t) + \frac{F}{r_1-r_2}u_{t_0}(t)\big[e^{r_1(t-t_0)} - e^{r_2(t-t_0)}\big] \\ &= \frac{y_1-r_2y_0}{r_1-r_2}e^{r_1t} + \frac{r_1y_0-y_1}{r_1-r_2}e^{r_2t} + \frac{F}{r_1-r_2}u_{t_0}(t)\big[e^{r_1(t-t_0)} - e^{r_2(t-t_0)}\big]. \end{aligned} \tag{7.21}$$

2. if $r^2+br+c=0$ has a double root r_1, then the solution y to (7.20) is

$$\begin{aligned} y(t) &= y_\infty(t) + Fu_{t_0}(t)(t-t_0)e^{r_1(t-t_0)} \\ &= y_0e^{r_1t} + (y_1-r_1y_0)te^{r_1t} + Fu_{t_0}(t)(t-t_0)e^{r_1(t-t_0)}. \end{aligned} \tag{7.22}$$

3. if $r^2+br+c=0$ has two complex roots $\lambda \pm i\mu$, then the solution y to (7.20) is

$$\begin{aligned} y(t) &= y_\infty(t) + \frac{F}{\mu}u_{t_0}(t)e^{\lambda(t-t_0)}\sin\mu(t-t_0) \\ &= y_0e^{\lambda t}\cos\mu t + \frac{y_1-\lambda y_0}{\mu}e^{\lambda t}\sin\mu t + \frac{F}{\mu}u_{t_0}(t)e^{\lambda(t-t_0)}\sin\mu(t-t_0). \end{aligned} \tag{7.23}$$

The uniform convergence of $\{y_\tau\}_{\tau>0}$ to y implies that if the support of the impulse is really small, even though we might not know the precise value of τ, the solution to (7.15) is very closed to the unique limit function y. We note that the three possible y's given above are continuous but have discontinuous derivatives, and are not differentiable at t_0.

7.5.1 The Dirac delta function

Even though we can stop our discussion about second order ODEs with impulse forcing functions here, we would like to go a little bit further by introducing the so-called "Dirac delta function". Taking (7.3) into account, (7.20) motivates the following

Definition 7.31 (Informal definition of the Dirac delta function)**.** For $t_0>0$, the ***Dirac delta function*** at t_0, denoted by δ_{t_0}, is the function whose Laplace transform is the function $G(s)=e^{-st_0}$.

Therefore, (7.3) and (7.20) imply that y satisfies the ODE

$$y''+by'+cy=F\delta_{t_0}(t), \qquad y(0)=y_0, \quad y'(0)=y_1. \tag{7.24}$$

By Theorem 7.19, in order to obtain the precise form of δ_{t_0} it suffices to find the function whose Laplace transform is the constant 1. However, this δ_{t_0} is **not** a function of non-negative real numbers since we actually have

$$y''(t)+by'(t)+cy(t)=0 \qquad \forall\, t\neq t_0$$

if y is given by (7.21), (7.22) or (7.22). If δ_{t_0} is a function of non-negative real numbers, no matter what value is assigned to $\delta_{t_0}(t_0)$, the Laplace transform of δ_{t_0} cannot be constant 1.

• **What does $y''+by'+cy = F\delta_{t_0}(t)$ really mean?** Recall that our goal is to find a "representative" of solutions of the sequence of ODEs (7.15). The discussion above shows that such a representative has to satisfies (7.20) which, under the assumption that

$$\mathscr{L}(y'' + by' + cy)(s) = (s^2 + bs + c)\mathscr{L}(y) - sy(0) - y'(0)\,. \tag{7.25}$$

implies the equation $y'' + by' + cy = F\delta_{t_0}(t)$. As we can see from the precise form of the function y in (7.21), (7.22) and (7.23), y' is not even continuous; thus (7.25) is in fact a false assumption.

The way that the ODE $y'' + by' + cy = F\delta_{t_0}(t)$ is understood is through the distribution theory, in which both sides of the ODE are treated as "functions of functions". Before our discussion, let us first have the following two definitions.

Definition 7.32. The collection of all k-times continuously differentiable function defined on $[0, \infty$ and vanishing outside some interval $[0, T]$ for some $T > 0$ is denoted by $\mathscr{C}_c^k([0,\infty))$. A function $f : [0, \infty) \to \mathbb{R}$ is said to belong to the space $\mathscr{C}_c^\infty([0,\infty))$ if $f \in \mathscr{C}_c^k([0,\infty))$ for all $k \in \mathbb{N}$. In other words,

$$\mathscr{C}_c^\infty([0,\infty)) \equiv \Big\{ f : [0,\infty) \to \mathbb{R} \,\Big|\, f \in \mathscr{C}_c^k([0,\infty)\ \forall\, k \in \mathbb{N} \Big\}\,.$$

Definition 7.33. Let $f : [0, \infty)$ be a piecewise continuous function. The linear functional induced by f, denoted by $\langle f, \cdot\rangle$, is a function on $\mathscr{C}_c^\infty([0,\infty))$ given by

$$\langle f, \varphi\rangle = \int_0^\infty f(t)\varphi(t)\,dt \qquad \forall\, \varphi \in \mathscr{C}_c^\infty([0,\infty))\,.$$

Consider the following simple ODE

$$y'' + by' + cy = f(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \tag{7.26}$$

where f is a continuous function of exponential order a for some $a \in \mathbb{R}$. The existence theory implies that there exists a unique twice continuously differentiable solution y to (7.26). Moreover, if $\varphi \in \mathscr{C}_c^2([0,\infty))$,

$$\int_0^\infty \big[y''(t) + by'(t) + cy(t)\big]\varphi(t)\,dt = \int_0^\infty f(t)\varphi(t)\,dt\,, \quad y(0) = y_0\,,\ y'(0) = y_1\,. \tag{7.27}$$

Since y is twice continuously differentiable on $[0, \infty)$, we can integrate by parts and find that the solution y to (7.26) also satisfies

$$y_0\varphi'(0) - \big(y_1 + by_0\big)\varphi(0) + \int_0^\infty y(t)\big(\varphi''(t) - b\varphi'(t) + c\varphi(t)\big)\,dt = \langle f, \varphi\rangle \quad \forall\, \varphi \in \mathscr{C}_c^2([0,\infty))\,. \tag{7.28}$$

On the other hand, if y is a twice continuously differentiable function satisfying (7.28), we can integrate by parts (to put the derivatives on φ back to y) and find that y satisfies

$$\begin{aligned}&\big(y_0 - y(0)\big)\varphi'(0) - \big[y_1 + by_0 - y'(0) - by(0)\big]\varphi(0)\\ &\quad + \int_0^\infty \big[y''(t) + by'(t) + cy(t)\big]\varphi(t)\,dt = \int_0^\infty f(t)\varphi(t)\,dt\end{aligned} \qquad \forall\, \varphi \in \mathscr{C}_c^2([0,\infty))\,.$$

In particular,

$$\int_0^\infty \big[y''(t)+by'(t)+cy(t)\big]\varphi(t)\,dt = \int_0^\infty f(t)\varphi(t)\,dt \quad \forall\,\varphi\in\mathscr{C}_c^2([0,\infty)) \text{ satisfying } \varphi(0)=\varphi'(0)=0\,.$$

Therefore, $y''+by'+cy$ must be identical to f since they are both continuous. Having established this, we find that

$$\big(y_0-y(0)\big)\varphi'(0)-\big[y_1+by_0-y'(0)-by(0)\big]\varphi(0)=0 \qquad \forall\,\varphi\in\mathscr{C}_c^2([0,\infty))\,.$$

Choose $\varphi\in\mathscr{C}_c^2([0,\infty))$ such that $\varphi(0)=0$ and $\varphi'(0)=1$, we conclude that $y_0=y(0)$; thus we arrive at the equality

$$\big[y_1+by_0-y'(0)-by(0)\big]\varphi(0)=0 \qquad \forall\,\varphi\in\mathscr{C}_c^2([0,\infty))\,.$$

The identity above clearly shows that $y_1=y'(0)$. In other words, if y is twice continuously differentiable and satisfies (7.28), then y satisfies (7.26); thus we establish that given a continuous forcing function f,

y is a solution to (7.26) if and only if y satisfies (7.28).

Thus we change the problem of solving an ODE "in the pointwise sense" to a problem of solving an integral equation which holds "in the sense of distribution" (a distribution means a function of functions). We note that there is one particular advantage of defining solution to (7.26) using (7.28) instead of (7.27): if f is discontinuous somewhere in $[0,\infty)$ (for example, $f=F\mathbf{1}_{[\alpha,\beta)}$ as in the previous section), (7.28) provides a good alternative even if y'' does not always exist.

The discussion above motivates the following

Definition 7.34 (Weak Solutions)**.** Let $f:[0,\infty)\to\mathbb{R}$ be a function of exponential order a for some $a\in\mathbb{R}$. A function $y:[0,\infty)\to\mathbb{R}$ is said to be a ***weak solution*** to (7.26) if y satisfies the integral equation (7.28). The integral equation (7.28) is called the ***weak formulation*** of (7.26).

We remark that the discussion above shows that if $f:[0,\infty)\to\mathbb{R}$ is continuous and of exponential order a for some $a\in\mathbb{R}$, the unique $\mathscr{C}^2$-solution y to (7.26) is also a weak solution.

In view of (7.28), if we define $L:\mathscr{C}_c^2([0,\infty))\to\mathbb{R}$ by

$$L(\varphi)=y_0\varphi'(0)-\big(y_1+by_0\big)\varphi(0)+\int_0^\infty y(t)\big(\varphi''(t)-b\varphi'(t)+c\varphi(t)\big)\,dt\,, \tag{7.29}$$

then the integral equation (7.26) is equivalent to that "the two linear functionals L and $\langle f,\cdot\rangle$ are the same on the space $\mathscr{C}_c^2([0,\infty))$". We also note that

$$L(\varphi)=\langle y''+by'+cy,\varphi\rangle \qquad \text{if } y'' \text{ is piecewise continuous, and } (y(0),y'(0))=(y_0,y_1)\,;$$

thus if y'' is piecewise continuous, the statement "$L=\langle f,\cdot\rangle$ on $\mathscr{C}_c^2([0,\infty))$" is the same as saying that "the linear functional induced by $y''+by'+cy$ and the linear functional induced by f are identical". This is what it means by $y''+by'+cy=f$ in the sense of distribution.

If the right-hand side $\langle f,\cdot\rangle$ is replaced by a general linear functional ℓ, we can still talk about the possibility of finding an integrable function y validating the integral equation (7.28), or more precisely, $L = \ell$ on $\mathscr{C}_c^2([0,\infty))$. In particular, for $F \in \mathbb{R}$ and $t_0 > 0$, it is reasonable to ask whether or not there exists an integrable function y such that

$$y_0\varphi'(0) - \big(y_1 + by_0\big)\varphi(0) + \int_0^\infty y(t)\big(\varphi''(t) - b\varphi'(t) + c\varphi(t)\big)\,dt = F\varphi(t_0) \quad \forall\,\varphi \in \mathscr{C}_c^2([0,\infty))\,, \tag{7.19}$$

where the linear functional $\ell : \mathscr{C}_c^2([0,\infty)) \to \mathbb{R}$ is given by

$$\ell(\varphi) = F\varphi(t_0) \qquad \forall\,\varphi \in \mathscr{C}_c^2([0,\infty))\,. \tag{7.30}$$

This is exactly the integral equation (7.19); thus the ODE $y'' + by' + cy = F\delta_{t_0}(t)$ is understood as $L = \ell$ on $\mathscr{C}_c^2([0,\infty))$, where L and ℓ are defined by (7.29) and (7.30), respectively.

The definition of ℓ motivates the following

Definition 7.35 (Dirac Delta Function)**.** For $t_0 \in \mathbb{R}$, let $\mathcal{X}(t_0)$ denote the collection of functions defined on $\mathbb{R}$ and continuous on an open interval containing t_0. The ***Dirac delta function*** at t_0 is a map $\delta_{t_0} : \mathcal{X}(t_0) \to \mathbb{R}$ defined by

$$\delta_{t_0}(\varphi) = \varphi(t_0)\,.$$

The map δ_0 is usually denoted as δ.

Under this definition, the ODE $y'' + by' + cy = F\delta_{t_0}$ is understood as "the functional induced by $y'' + by' + cy$ (given by (7.29)) is the same as the functional induced by $F\delta_{t_0}$". The function y given by (7.21), (7.22) or (7.23) is then a ***weak solution*** to (7.24).

Example 7.36. In this example, we would like to find the "anti-derivative" of the Dirac delta function at $t_0 > 0$. In other words, we are looking for a solution to

$$y' = \delta_{t_0}(t)\,, \qquad y(0) = 0\,, \quad y'(0) = 0\,.$$

Taking the Laplace transform, we find that

$$s\mathscr{L}(y)(s) = e^{-st_0} \quad \text{or equivalently,} \quad \mathscr{L}(y)(s) = \frac{e^{-st_0}}{s}\,. \tag{7.31}$$

As a consequence, by Example 7.5 and Theorem 7.19 we conclude that the (weak) solution to the ODE above is

$$y(t) = u_{t_0}(t) = H(t - t_0)\,.$$

We again emphasize that in principle we are not allowed to use Theorem 7.12 or Corollary 7.13 to compute the Laplace transform of y'; however, the functional induced by y' (by assuming that y is

$$\int_0^\infty y'(t)\varphi(t)\,dt = y(0)\varphi(0) - \int_0^\infty y(t)\varphi'(t)\,dt$$

so we are in fact solving $y' = \delta_{t_0}(t)$ in the sense of distribution; that is, we look for y satisfying

$$-\int_0^\infty y(t)\varphi'(t)\,dt = \varphi(t_0) \qquad \forall\,\varphi \in X.$$

Letting $\varphi(t) = e^{-st}$ leads to (7.31).

7.6 The Convolution Integrals

Definition 7.37. Let f, g be piecewise continuous on $[0,\infty)$. The ***convolution*** of f and g, denoted by $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau\,. \tag{7.32}$$

Proposition 7.38. *Let f, g, h be piecewise continuous on $[0,\infty)$. Then*

(a) $f * g = g * f$;

(b) $f * (g+h) = (f * g) + (f * h)$;

(c) $(f * g) * h = f * (g * h)$;

(d) $(f * 0) = 0$.

Theorem 7.39. *Let f and g be piecewise continuous on $[0,\infty)$ and are of exponential order a. Then*

$$\mathscr{L}(f * g)(s) = \mathscr{L}(f)(s)\mathscr{L}(g)(s) \qquad \forall\, s > a\,.$$

Proof. Since f is of exponential order a, for some $M_1 > 0$, $|f(t)| \leqslant M_1 e^{at}$ for all $t > 0$. Therefore, for $s > a$,

$$\Big|\mathscr{L}(f)(s) - \int_0^R e^{-st} f(t)\,dt\Big| \leqslant \int_R^\infty e^{-st}\big|f(t)\big|\,dt \leqslant M_1 \int_R^\infty e^{-(s-a)t} dt \leqslant \frac{M_1}{s-a} e^{(a-s)R}\,.$$

Similarly, for some $M_2 > 0$, $|g(t)| \leqslant M_2 e^{at}$ for all $t > 0$ and

$$\Big|\mathscr{L}(g)(s) - \int_0^R e^{-st} g(t)\,dt\Big| \leqslant \frac{M_2}{s-a} e^{(a-s)R} \qquad \forall\, s > a\,.$$

By the Fubini theorem,

$$\begin{aligned}\int_0^R e^{-st}\Big(\int_0^t f(t-\tau)g(\tau)\,d\tau\Big)\,dt &= \int_0^R \Big(\int_\tau^R f(t-\tau)g(\tau)e^{-st}\,dt\Big)\,d\tau \\ &= \int_0^R e^{-s\tau} g(\tau)\Big(\int_\tau^R f(t-\tau)e^{-s(t-\tau)}\,dt\Big)\,d\tau = \int_0^R e^{-s\tau} g(\tau)\Big(\int_0^{R-\tau} f(t)e^{-st}\,dt\Big)\,d\tau\,;\end{aligned}$$

thus for $s > a$,

$$\begin{aligned}&\Big|\int_0^R e^{-st}\Big(\int_0^t f(t-\tau)g(\tau)\,d\tau\Big)\,dt - \mathscr{L}(f)(s)\mathscr{L}(g)(s)\Big| \\ &\quad = \Big|\int_0^R e^{-s\tau} g(\tau)\Big(\int_0^{R-\tau} f(t)e^{-st}\,dt\Big)\,d\tau - \mathscr{L}(f)(s)\mathscr{L}(g)(s)\Big| \\ &\quad = \Big|\int_0^R e^{-s\tau} g(\tau)\Big(\int_0^{R-\tau} f(t)e^{-st}\,dt - \mathscr{L}(f)(s)\Big)\,d\tau + \mathscr{L}(f)(s)\Big(\int_0^R e^{-s\tau} g(\tau)\,d\tau - \mathscr{L}(g)(s)\Big)\Big| \\ &\quad \leqslant \frac{M_1 M_2}{s-a}\int_0^R e^{-s\tau} e^{a\tau} e^{(a-s)(R-\tau)}\,d\tau + \frac{M_2}{s-a}\big|\mathscr{L}(f)(s)\big| e^{(a-s)R} \\ &\quad = \frac{M_1 M_2}{s-a} R e^{(a-s)R} + \frac{M_2}{s-a}\big|\mathscr{L}(f)(s)\big| e^{(a-s)R}\end{aligned}$$

which converges to 0 as $R \to \infty$. ◻

Example 7.40. Find the inverse Laplace transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)}$.

Method 1: Using the partial fractions,

$$\frac{a}{s^2(s^2+a^2)} = \frac{1}{a}\Big[\frac{1}{s^2} - \frac{1}{s^2+a^2}\Big] = \frac{1}{a}\cdot\frac{1}{s^2} - \frac{1}{a^2}\frac{a}{s^2+a^2};$$

thus Example 7.7 and 7.9 imply

$$\mathscr{L}^{-1}(H)(t) = \frac{t}{a} - \frac{1}{a^2}\sin at\,.$$

Method 2: By Theorem 7.39, with F, G denoting the functions $F(s) = \dfrac{1}{s^2}$ and $G(s) = \dfrac{a}{s^2+a^2}$,

$$\begin{aligned}
\mathscr{L}^{-1}(H)(t) &= \big(\mathscr{L}^{-1}(F) * \mathscr{L}^{-1}(G)\big)(t) = \int_0^t (t-\tau)\sin(a\tau)\,d\tau \\
&= t\int_0^t \sin a\tau\,d\tau - \int_0^t \tau\sin a\tau\,d\tau \\
&= -\frac{t}{a}\cos(a\tau)\Big|_{\tau=0}^{\tau=t} - \Big[-\frac{\tau}{a}\cos(a\tau)\Big|_{\tau=0}^{\tau=t} + \frac{1}{a}\int_0^t \cos(a\tau)\,d\tau\Big] \\
&= \frac{t}{a} - \frac{1}{a}\int_0^t \cos(a\tau)\,d\tau = \frac{t}{a} - \frac{\sin a\tau}{a^2}\Big|_{\tau=0}^{\tau=t} = \frac{t}{a} - \frac{\sin at}{a^2}\,.
\end{aligned}$$

Example 7.41. Find the (weak) solution of the initial value problem

$$y'' + 4y = g(t)\,, \qquad y(0) = 3\,, \quad y'(0) = -1\,.$$

Taking the Laplace transform of the equation above, we find that

$$\mathscr{L}(y)(s) = \frac{3s-1}{s^2+4} + \frac{\mathscr{L}(g)(s)}{s^2+4} = \frac{3s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{\mathscr{L}(g)(s)}{2}\frac{2}{s^2+4}\,.$$

Therefore, by Example 7.7 and Theorem 7.39,

$$\begin{aligned}
y(t) &= 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t g(t-\tau)\sin 2\tau\,d\tau \\
&= 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t g(\tau)\sin 2(t-\tau)\,d\tau\,.
\end{aligned}$$

In general, we can consider the second order ODE

$$y'' + by' + cy = g(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

As discussed before, we find that if y is a (weak) solution to the ODE above,

$$\mathscr{L}(y)(s) = \frac{(s+b)y_0 + y_1}{s^2+bs+c} + \frac{\mathscr{L}(g)(s)}{s^2+bs+c}\,.$$

Therefore,

1. if $r^2 + br + c = 0$ has two distinct real roots r_1 and r_2, then the solution y is

$$y(t) = \frac{y_1 - r_2 y_0}{r_1 - r_2} e^{r_1 t} + \frac{r_1 y_0 - y_1}{r_1 - r_2} e^{r_2 t} + \int_0^t g(t-\tau) \frac{e^{r_1\tau} - e^{r_2\tau}}{r_1 - r_2}\, d\tau\,.$$

2. if $r^2 + br + c = 0$ has a double root r_1, then the solution y is

$$y(t) = y_0 e^{r_1 t} + (y_1 - r_1 y_0) t e^{r_1 t} + \int_0^t g(t-\tau) e^{r_1\tau} \tau\, d\tau\,.$$

3. if $r^2 + br + c = 0$ has two complex roots $\lambda \pm i\mu$, then the solution y is

$$y(t) = y_0 e^{\lambda t} \cos \mu t + \frac{y_1 - \lambda y_0}{\mu} e^{\lambda t} \sin \mu t + \int_0^t g(t-\tau) e^{\lambda\tau} \frac{\sin \mu\tau}{\mu}\, d\tau\,.$$

8 Partial Differential Equations and Fourier Series

8.1 Two-Point Boundary Value Problems

For a second order ODE $y'' + p(t)y' + q(t)y = g(t)$, instead of imposing the initial condition $y(t_0) = y_0$ and $y'(t_0) = y_1$ sometimes the boundary condition $y(\alpha) = y_0$ and $y(\beta) = y_1$ are imposed. In this section, we consider the two-point boundary value problem

$$y'' + p(x)y' + q(x)y = g(x)\,, \qquad y(\alpha) = y_0, \quad y(\beta) = y_1\,. \tag{8.1}$$

Let $z(x) = y(x) - \dfrac{x-\alpha}{\beta-\alpha} y_1 - \dfrac{x-\beta}{\alpha-\beta} y_0$. Then z satisfies

$$z'' + p(x)z' + q(x)z = G(x)\,, \qquad z(\alpha) = z(\beta) = 0\,, \tag{8.2}$$

where $G(x) = g(x) - p(x)\dfrac{y_0 - y_1}{\alpha - \beta} - q(x)\big(\dfrac{x-\alpha}{\beta-\alpha} y_1 + \dfrac{x-\beta}{\alpha-\beta} y_0\big)$. Therefore, in general we can assume the homogeneous boundary condition $y_0 = y_1 = 0$ in (8.1). Such a boundary condition is called ***homogeneous Dirichlet*** boundary condition, while the boundary condition in (8.1) is called ***in-homogeneous Dirichlet*** boundary conditions.

Remark 8.1. Even though the initial value problem

$$y'' + p(t)y' + q(t)y = g(t)\,, \qquad y(t_0) = y_0\,, \quad y'(t_0) = y_1 \tag{8.3}$$

looks quite similar to the boundary value problem (8.1), they actually differ in some very important ways. For example, if p, q, g are continuous, the initial value problem (8.3) always have a unique solution, while the boundary value problem (8.1) might have no solution or infinitely many solutions:

1. $y'' + y = 0$ with boundary condition $y(0) = y(\pi) = 0$ has infinite many solutions $y_c(x) = c \sin x$.

2. $y'' + y = \sin x$ with boundary condition $y(0) = y(\pi) = 0$ has no solution. To see this, we assume that there is a solution $y = y(x)$ to this ODE. Then $y'(0) = y_1$ for some $y_1 \in \mathbb{R}$. Use the Laplace transform (treating x as the variable t), we find that the solution y satisfies

$$\mathscr{L}(y)(s) = \frac{y_1}{s^2+1} + \frac{1}{(s^2+1)^2}\,;$$

thus by Theorem 7.39 we find that

$$y(x) = y_1 \sin x + \int_0^x \sin(x-z)\sin z\, dz = y_1 \sin x + \frac{\sin x - x\cos x}{2}.$$

It is impossible to have $y(\pi) = 0$ for any choice of y_1.

On the other hand, there are cases that (8.1) has a unique solution. For example, the general solution to the boundary value problem

$$y'' + 2y = 0$$

is given by

$$y(x) = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x\,;$$

thus to validate the boundary condition $y(0) = 1$ and $y(\pi) = 0$, we must have $C_1 = 1$ and $C_2 = -\cot\sqrt{2}\pi$. In other words, the solution $y(x) = \cos\sqrt{2}x - \cot\sqrt{2}\pi \sin\sqrt{2}x$.

The existence theory of the solution to (8.1) requires a totally different functional framework, and will not be proved in this course. However, we will still state the existence theory and try to explain the idea of why the theorem should be true.

Theorem 8.2. *Let α, β be real numbers and $\alpha < \beta$. Suppose that $p : [\alpha, \beta] \to \mathbb{R}$ is continuously differentiable, and $q : [\alpha, \beta] \to \mathbb{R}$ is continuous. Then* (8.1) (*with* $y_0 = y_1 = 0$) *has a solution if and only if $g : [\alpha, \beta] \to \mathbb{R}$ is integrable and*

$$\int_\alpha^\beta g(x)\varphi(x)\, dx = 0 \quad \forall\, \varphi \text{ satisfying } \varphi'' - p(x)\varphi' + \big(q(x) - p'(x)\big)\varphi = 0 \text{ and } \varphi(\alpha) = \varphi(\beta) = 0\,.$$

The solution is unique if the ODE $y'' + p(x)y' + q(x)y = 0$ with $y(\alpha) = y(\beta) = 0$ has only trivial solution $y \equiv 0$.

Remark 8.3. The equation $\varphi'' - p(x)\varphi' + \big(q(x) - p'(x)\big)\varphi = 0$ is called the ***formal adjoint equation*** of $y'' + p(x)y' + q(x)y = 0$.

Example 8.4. Consider $y'' + y = g(x)$ with boundary data $y(0) = y(\pi) = 0$, where $g(x) = \sin x$. We have shown in Remark 8.1 that there is no solution to this boundary value problem. To see this using Theorem 8.2, we first find the kernel of the formal adjoint equation

$$\varphi'' + \varphi = 0\,, \qquad \varphi(0) = \varphi(\pi) = 0\,.$$

Since the general solution to $\varphi'' + \varphi = 0$ is $\varphi(t) = C_1 \cos x + C_2 \sin x$, to validate the boundary data we must have $C_1 = 0$. Therefore, for the ODE under consideration to have a solution, we must have

$$\int_0^\pi g(x) \sin x\, dx = 0\,.$$

This is impossible since $g(x) = \sin x$.

Example 8.5. Again consider $y'' + y = g(x)$ with boundary data $y(0) = y(\pi) = 0$, but this time we let $g(x) = \cos x$. As discussed above, since

$$\int_0^\pi g(x)\sin x\,dx = \int_0^\pi \sin x\cos x\,dx = \frac{1}{2}\int_0^\pi \sin 2x\,dx = \frac{-\cos 2x}{4}\Big|_{x=0}^{x=\pi} = 0\,,$$

by Theorem 8.2 this ODE has a solution.

To find a solution to the ODE above, we mimic the procedure in Remark 8.1 and find that

$$\mathscr{L}(y)(s) = \frac{y_1}{s^2+1} + \frac{s}{(s^2+1)^2}\,,$$

where $y_1 = y'(0)$. Therefore, Theorem 7.39 implies that

$$y(x) = y_1 \sin x + \int_0^x \cos(x-z)\sin z\,dz = y_1\sin x + \frac{x\sin x}{2}\,.$$

Reason/Idea for why Theorem 8.2 is true: Suppose that A is a $n\times n$ matrix, $\boldsymbol{b}\in\mathbb{R}^n$. Then $\mathbb{R}^n = \mathrm{R}(A)\oplus\mathrm{Ker}(A^{\mathrm{T}})$, and $\mathrm{R}(A)\perp\mathrm{Ker}(A^{\mathrm{T}})$; that is,

$$\boldsymbol{x}\cdot\boldsymbol{y} = 0 \qquad \forall\,\boldsymbol{x}\in\mathrm{R}(A) \text{ and } \boldsymbol{y}\in\mathrm{Ker}(A^{\mathrm{T}})\,.$$

Therefore,

$$\boldsymbol{b}\in\mathrm{R}(A) \qquad \text{if and only if} \qquad \boldsymbol{b}\cdot\boldsymbol{y} = 0 \quad \forall\,\boldsymbol{y}\in\mathrm{Ker}(A^{\mathrm{T}})\,.$$

Now, we treat

1. the differential operator $\dfrac{d^2}{dx^2} + p(x)\dfrac{d}{dx} + q(x)$ as the role of A;
2. the space of twice differentiable functions with vanishing boundary data as the role of $\mathbb{R}^n$;
3. the integral over $[\alpha,\beta]$ of product of functions f, g as the inner product of f and g.

Then conceptually we can expect that

$$g\in\mathrm{R}(A) \quad \text{if and only if} \quad g\cdot\varphi = 0 \quad \forall\,\varphi\in\mathrm{Ker}(A^{\mathrm{T}})\,. \tag{8.4}$$

Now let us examine what $\mathrm{Ker}(A^{\mathrm{T}})$ is. By definition, A^{T} is the unique operator satisfying $(A\boldsymbol{x})\cdot\boldsymbol{y} = \boldsymbol{x}\cdot(A^{\mathrm{T}}\boldsymbol{y})$; thus for $y,z\in\mathrm{Dom}(A)$ (which is the collection of twice differentiable functions with vanishing boundary data) A^{T} has the property that

$$\int_\alpha^\beta (Ay)(x)z(x) = \int_\alpha^\beta y(x)(A^{\mathrm{T}}z)(x)\,dx\,.$$

Integrating by parts, by the fact that $y(\alpha) = y(\beta) = z(\alpha) = z(\beta) = 0$,

$$\begin{aligned}\int_\alpha^\beta (Ay)(x)z(x) &= \int_\alpha^\beta \big[y'' + p(x)y' + q(x)y\big]z(x)\,dx = \int_\alpha^\beta y(x)\big[z'' - \big(p(x)z\big)' + q(x)z\big]\,dx\\ &= \int_\alpha^\beta y(x)(A^{\mathrm{T}}z)(x)\,dx\,.\end{aligned}$$

Therefore, A^{T} is the differential operator $\dfrac{d^2}{dx^2} - p(x)\dfrac{d}{dx} + \big(q(x) - p'(x)\big)$. Note that $g \in \mathrm{R}(A)$ means

$$\exists\, y \in \mathrm{Dom}(A) \ni y'' + p(x)y' + q(x)y = g(x)\,;$$

thus (8.4) implies that

$$\begin{aligned} & \exists\, y \in \mathrm{Dom}(A) \ni y'' + p(x)y' + q(x)y = g(x) \\ \Leftrightarrow \quad & \int_\alpha^\beta g(x)\varphi(x)\,dx = 0 \quad \forall\, \varphi \text{ satisfying } \varphi'' - p(x)\varphi' + \big(q(x) - p'(x)\big)\varphi = 0\,. \end{aligned}$$

This is exactly what Theorem 8.2 is talking about. However, we emphasize that the argument above is purely conceptually but not rigorous.

8.1.1 Eigenfunctions

Recall that if A is a real symmetric $n \times n$ matrix, then it is diagonalizable and there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A. Similarly, if a second order differential operator

$$A = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) \tag{8.5}$$

is self-adjoint (meaning $A = A^{\mathrm{T}}$, where A^{T} is given by $A^{\mathrm{T}}\varphi = \varphi'' - p(x)\varphi' + (q(x) - p'(x))\varphi$), then the eigenvectors of A, called the ***eigenfunctions*** of A, can also form an orthonormal basis of $\mathrm{Dom}(A)$. We note that for a differential operator A given by (8.5) being self-adjoint, it is sufficient and necessary that $p \equiv 0$. In particular, we consider the eigenfunctions u of the differential operator $\Delta = \dfrac{d^2}{dx^2}$ satisfying

$$\Delta u = u'' = \lambda u\,, \qquad u(\alpha) = u(\beta) = 0 \quad (\alpha < \beta)\,. \tag{8.6}$$

If $\lambda > 0$, then the general solution to (8.6) is $u(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$ which, to validate the boundary data, implies that $C_1 = C_2 = 0$. Therefore, the eigenvalue of the differential operator Δ cannot be positive.

If $\lambda \leqslant 0$, the general solution to (8.6) is $u(x) = C_1 \cos\sqrt{-\lambda}x + C_2 \sin\sqrt{-\lambda}x$. To satisfy the boundary data, it is required that

$$\begin{bmatrix} \cos\sqrt{-\lambda}\alpha & \sin\sqrt{-\lambda}\alpha \\ \cos\sqrt{-\lambda}\beta & \sin\sqrt{-\lambda}\beta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since we are interested in the case that C_1 or $C_2 \neq 0$, we must have

$$\det\left(\begin{bmatrix} \cos\sqrt{-\lambda}\alpha & \sin\sqrt{-\lambda}\alpha \\ \cos\sqrt{-\lambda}\beta & \sin\sqrt{-\lambda}\beta \end{bmatrix} \right) = 0\,;$$

thus $\sin\sqrt{-\lambda}(\beta - \alpha) = 0$. This implies that $\sqrt{-\lambda}(\beta - \alpha) = k\pi$ or some k; thus $\lambda = -\dfrac{k^2\pi^2}{(\beta - \alpha)^2}$. A corresponding eigenfunction is then

$$u(x) = -\sin\frac{k\pi\alpha}{\beta - \alpha}\cos\frac{k\pi x}{\beta - \alpha} + \cos\frac{k\pi\alpha}{\beta - \alpha}\sin\frac{k\pi x}{\beta - \alpha} = \sin\frac{k\pi(x - \alpha)}{\beta - \alpha}\,.$$

For each $k \in \mathbb{N}$, define

$$\lambda_k = -\frac{k^2\pi^2}{(\beta-\alpha)^2} \quad \text{and} \quad \mathrm{e}_k(x) = \sqrt{\frac{2}{\beta-\alpha}} \sin\frac{k\pi(x-\alpha)}{\beta-\alpha}\,, \tag{8.7}$$

where the constant $\sqrt{\frac{2}{\beta-\alpha}}$ is for the purpose that $\{\mathrm{e}_k\}_{k=1}^{\infty}$ forms an orthonormal set; that is, we have the property that $\mathrm{e}_k \cdot \mathrm{e}_j = \delta_{kj}$, or to be more precisely,

$$\int_\alpha^\beta \mathrm{e}_k(x)\mathrm{e}_j(x)\,dx = \begin{cases} 1 & \text{if } k = j\,, \\ 0 & \text{if } k \neq j\,. \end{cases}$$

Then we expect that for twice differentiable function φ with vanishing Dirichlet boundary data,

$$\varphi(x) = \sum_{k=1}^{\infty}(\varphi\cdot\mathrm{e}_k)\mathrm{e}_k(x) = \sum_{k=1}^{\infty}\int_\alpha^\beta \varphi(y)\mathrm{e}_k(y)\,dy\,\mathrm{e}_k(x) \qquad \forall\, x \in [\alpha,\beta] \tag{8.8}$$

and the "length" of the function φ should obey the Pythagorean Theorem; that is, one expects that

$$\int_\alpha^\beta \varphi(x)^2\,dx = \sum_{k=1}^{\infty}\Big(\int_\alpha^\beta \varphi(y)\mathrm{e}_k(y)\,dy\Big)^2 \tag{8.9}$$

Identity (8.8), often called the ***Fourier series representation*** (for functions vanishing on the boundary), in fact holds for all φ which satisfies $\varphi(\alpha) = \varphi(\beta)$ and is Hölder continuous with some Hölder exponent; that is, there is $\alpha \in (0,1]$ such that

$$\sup_{x_1,x_2\in[\alpha,\beta]} \frac{\big|\varphi(x_1)-\varphi(x_2)\big|}{|x_1-x_2|^\alpha} < \infty\,,$$

while (8.9), called the ***Parseval identity***, even holds for a larger class of functions. We again emphasize that the derivation of (8.8) is not rigorous but purely conceptually.

Instead of considering the second order equation $y'' + p(t)y' + q(t)y = g(t)$ with boundary $y(\alpha) = y(\beta) = 0$, we can also consider the following three type of boundary conditions:

1. $y'(\alpha) = a$, $y'(\beta) = b$, called the ***inhomogeneous Neumann*** boundary condition, or

2. $y(\alpha) = 0$, $y'(\beta) = b$ or $y'(\alpha) = a$, $y'(\beta) = 0$, called the ***mixed type*** boundary condition.

We note that in either cases, using similar technique to transform (8.1) to (8.2) we can always transform the boundary condition above to the homogeneous one; that is,

1. $y'(\alpha) = 0$, $y'(\beta) = 0$, called the ***homogeneous Neumann*** boundary condition, or

2. $y(\alpha) = 0$, $y'(\beta) = 0$ or $y'(\alpha) = 0$, $y'(\beta) = 0$.

Now we consider the eigenfunctions for the differential operator Δ with different boundary conditions.

1. **Homogeneous Neumann boundary conditions**: We look for $u : [\alpha,\beta] \to \mathbb{R}$ satisfying

$$u_{xx} = \lambda u \qquad \text{in} \quad [\alpha,\beta], \quad u'(\alpha) = u'(\beta) = 0\,.$$

As in the previous section, if $\lambda > 0$, then the only possible u is trivial, so we consider the case $\lambda \leqslant 0$. If $\lambda = 0$, we have $u(x) = 1$ being a non-trivial eigenfunction. If $\lambda < 0$, the general solution to the ODE (without specifying the boundary condition) is

$$u(x) = C_1 \cos \sqrt{-\lambda} x + C_2 \sin \sqrt{-\lambda} x\,,$$

and to validate the boundary condition, the system

$$\begin{bmatrix} -\sin \sqrt{-\lambda}\alpha & \cos \sqrt{-\lambda}\alpha \\ -\sin \sqrt{-\lambda}\beta & \cos \sqrt{-\lambda}\beta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

must have non-trivial solution which implies $\sin \sqrt{-\lambda}(\beta - \alpha) = 0$. As in the previous section, we conclude that

$$\lambda = -\frac{k^2\pi^2}{(\beta-\alpha)^2} \quad \text{and} \quad u(x) = C \cos \sqrt{-\lambda}(x - \alpha) = C \cos \frac{k\pi(x-\alpha)}{\beta - \alpha}\,.$$

For each $k \in \mathbb{N}$, define

$$\lambda_k = -\frac{k^2\pi^2}{(\beta-\alpha)^2}\,, \quad \mathrm{e}_0(x) = \sqrt{\frac{1}{\beta-\alpha}} \quad \text{and} \quad \mathrm{e}_k(x) = \sqrt{\frac{2}{\beta-\alpha}} \cos \frac{k\pi(x-\alpha)}{\beta-\alpha}\,. \tag{8.10}$$

Then $\{\mathrm{e}_k\}_{k=0}^{\infty}$ forms an orthonormal "basis" in the space

$$\big\{u \in \mathscr{C}^2([\alpha, \beta]) \,\big|\, u'(\alpha) = u'(\beta) = 0\big\}\,;$$

that is, for twice differentiable function φ with vanishing Neumann boundary data,

$$\varphi(x) = \sum_{k=0}^{\infty} (\varphi \cdot \mathrm{e}_k)\mathrm{e}_k(x) = \sum_{k=1}^{\infty} \int_\alpha^\beta \varphi(y)\mathrm{e}_k(y)\,dy\,\mathrm{e}_k(x) \qquad \forall\, x \in (\alpha, \beta) \tag{8.11}$$

and the "length" of the function φ should obey the Parseval identity

$$\int_\alpha^\beta \varphi(x)^2\,dx = \sum_{k=1}^{\infty} \Big(\int_\alpha^\beta \varphi(y)\mathrm{e}_k(y)\,dy\Big)^2 \tag{8.12}$$

2. **Mixed type boundary conditions**: We first look for $u : [\alpha, \beta] \to \mathbb{R}$ satisfying

$$u_{xx} = \lambda u \qquad \text{in} \quad [\alpha, \beta], \quad u(\alpha) = u'(\beta) = 0\,.$$

Similar computations show that $\lambda < 0$ and $\cos \sqrt{-\lambda}(\alpha - \beta) = 0$. Therefore, $\sqrt{-\lambda}(\beta - \alpha) = \dfrac{(2k+1)\pi}{2}$ which implies that for some $k \in \mathbb{N}$,

$$\lambda = -\frac{(2k-1)^2\pi^2}{4(\beta-\alpha)^2}\,.$$

Therefore, for each $k \in \mathbb{N}$, define

$$\lambda_k = -\frac{(2k-1)^2\pi^2}{4(\beta-\alpha)^2} \quad \text{and} \quad \mathrm{e}_k(x) = \sqrt{\frac{2}{\beta-\alpha}} \sin \frac{(2k-1)\pi(x-\alpha)}{2(\beta-\alpha)}\,.$$

Then $\{\mathrm{e}_k\}_{k=1}^{\infty}$ forms an orthonormal "basis" in the space $\big\{u \in \mathscr{C}^2([\alpha, \beta]) \,\big|\, u(\alpha) = u'(\beta) = 0\big\}$. Similarly, $\Big\{\sqrt{\frac{2}{\beta-\alpha}} \cos \frac{(2k-1)\pi(x-\alpha)}{2(\beta-\alpha)}\Big\}_{k=1}^{\infty}$ forms an orthonormal "basis" in the space $\big\{u \in \mathscr{C}^2([\alpha, \beta]) \,\big|\, u'(\alpha) = u(\beta) = 0\big\}$.

8.2 Fourier Series

In the previous section, we discuss how one obtain an orthonormal basis in different spaces. In fact, by the Stone-Weierstrass Theorem (abstract version) we can conclude the following

Theorem 8.6. *Let $\mathscr{C}(\mathbb{T})$ be the collection of all 2π-periodic continuous functions, and $\mathscr{P}_n(\mathbb{T})$ be the collection of all trigonometric polynomials of degree n; that is,*

$$\mathscr{P}_n(\mathbb{T}) = \left\{ \frac{c_0}{2} + \sum_{k=1}^{n} c_k \cos kx + s_k \sin kx \,\middle|\, \{c_k\}_{k=0}^n, \{s_k\}_{k=1}^n \subseteq \mathbb{R} \right\}.$$

Let $\mathscr{P}(\mathbb{T}) = \bigcup\limits_{n=0}^{\infty} \mathscr{P}_n(\mathbb{T})$. Then $\mathscr{P}(\mathbb{T})$ is dense in $\mathscr{C}(\mathbb{T})$. In other words, if $f \in \mathscr{C}(\mathbb{T})$ and $\varepsilon > 0$ is given, there exists $p \in \mathscr{P}(\mathbb{T})$ such that

$$\big|f(x) - p(x)\big| < \varepsilon \qquad \forall\, x \in \mathbb{R}.$$

In other words, every period function with period 2π can be approximated by trigonometric polynomials in the uniform sense. In this section, we would like to discuss how to approximate a continuous period functions using trigonometric polynomials.

背景知識：Stone-Weierstrass 定理（concrete version）告訴我們定義在 $[0,1]$ 上的連續函數 f 可以用多項式（例如 Bernstein 多項式）去逼近（在均勻收斂的意義下），而我們也注意到 Bernstein 多項式，在取不同次數 n 的多項式做逼近時，每一個單項式 x^k 前面的係數都跟 n 和 k 有關。但是從 Taylor 定理中我們又發現，對某些擁有很好性質的函數 f（叫做解析函數 Analytic functions），即使取不同次數 n 的多項式做逼近時，每個單項式 x^k 前面的係數可以取成只跟函數 f 的 k 次導數有關（跟 n 無關）。這給了我們一個很粗略的概念，知道想用多項式去逼近連續函數時，多項式的係數有些時候會跟多項次的次數有關，有時則無關。

在這一節中，我們特別關注連續的週期函數。由 Theorem 8.6 我們知道週期為 2π 的函數可用形如

$$p_n(x) = \frac{c_0^{(n)}}{2} + \sum_{k=1}^{n} (c_k^{(n)} \cos kx + s_k^{(n)} \sin kx)$$

的三角多項式 (trigonometric polynomials) 所逼近（在均勻收斂的意義下），其中上標 (n) 代表的是係數可能與用來逼近的三角多項式的次數 n 有關係。跟前一段所述的經驗類似，在數學理論上我們想知道下面問題的答案：

1. 什麼樣的函數，可以用係數與逼近次數無關的三角多項式去逼近。對這樣的函數，三角多項式要怎麼挑？

2. 對於實在沒辦法用係數與逼近次數無關的三角多項式去逼近的連續週期函數，有什麼好的方法逼近？而上面所挑出來的那個係數跟逼近次數無關的三角多項式，在次數接近無窮大時出了什麼問題？

Let $f \in \mathscr{C}(\mathbb{T})$ be given. We first assume that the trigonometric polynomials used to approximate f can be chosen in such a way that the coefficients does not depend on the degree of approximation;

that is, $c_k^{(n)} = c_k$ and $s_k^{(n)} = s_k$. In this case, if $p_n \to f$ uniformy on $[-\pi, \pi]$, we must have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} p_n(x) \cos kx \, dx = \int_{-\pi}^{\pi} f(x) \cos kx \, dx \qquad \forall\, k \in \{0, 1, \cdots, n\}$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} p_n(x) \sin kx \, dx = \int_{-\pi}^{\pi} f(x) \sin kx \, dx \qquad \forall\, k \in \{1, \cdots, n\}\,.$$

Since

$$\int_{-\pi}^{\pi} \cos kx \cos \ell x \, dx = \int_{-\pi}^{\pi} \sin kx \sin \ell x \, dx = \pi\delta_{k\ell} \qquad \forall\, k, \ell \in \mathbb{N}$$

and

$$\int_{-\pi}^{\pi} \sin kx \cos \ell x \, dx = 0 \qquad \forall\, k \in \mathbb{N}, \ell \in \mathbb{N} \cup \{0\}\,,$$

we find that

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \quad \text{and} \quad s_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx\,. \tag{8.13}$$

This induces the following

Definition 8.7. For a Riemann integrable function $f : [-\pi, \pi] \to \mathbb{R}$, the ***Fourier series representation*** of f, denoted by $s(f, \cdot)$, is given by

$$s(f, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} (c_k \cos kx + s_k \sin kx)$$

whenever the sum makes sense, where sequences $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ given by (8.13) are called the ***Fourier coefficients*** associated with f. The n-th partial sum of the Fourier series representation to f, denoted by $s_n(f, \cdot)$, is given by

$$s_n(f, x) = \frac{c_0}{2} + \sum_{k=1}^{n} (c_k \cos kx + s_k \sin kx)\,.$$

We note that for the Fourier series $s(f, x)$ to be defined, f is not necessary continuous.

Example 8.8. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \pi\,, \\ -x & \text{if } -\pi < x < 0\,, \end{cases}$$

and $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. To find the Fourier representation of f, we compute the Fourier coefficients by

$$s_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = \frac{1}{\pi}\Big(\int_0^{\pi} x \sin kx \, dx - \int_{-\pi}^{0} x \sin kx \, dx \Big) = 0$$

and

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \frac{1}{\pi}\Big(\int_0^{\pi} x \cos kx \, dx - \int_{-\pi}^{0} x \cos kx \, dx \Big) = \frac{2}{\pi} \int_0^{\pi} x \cos kx \, dx\,.$$

If $k=0$, then $c_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi x\,dx = \pi$, while if $k \in \mathbb{N}$,

$$c_k = \frac{2}{\pi}\Big(\frac{x\sin kx}{k}\Big|_0^\pi - \int_0^\pi \frac{\sin kx}{k}\,dx\Big) = \frac{2}{\pi}\frac{\cos kx}{k^2}\Big|_0^\pi = \frac{2((-1)^k-1)}{\pi k^2}\,.$$

Therefore, $c_{2k}=0$ and $c_{2k-1} = -\dfrac{4}{\pi(2k-1)^2}$ for all $k \in \mathbb{N}$. Therefore, the Fourier series representation of f is given by

$$s(f,x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^\infty \frac{\cos(2k-1)x}{(2k-1)^2}\,.$$

Example 8.9. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } -\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}\,, \\ 0 & \text{if } -\pi \leqslant x < -\dfrac{\pi}{2} \text{ or } \dfrac{\pi}{2} < x \leqslant \pi\,, \end{cases}$$

and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. We compute the Fourier coefficients of f and find that $s_k = 0$ for all $k \in \mathbb{N}$ and $c_0 = 1$, as well as

$$c_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos kx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos kx\,dx = \frac{2\sin\frac{k\pi}{2}}{\pi k}\,.$$

Therefore, $c_{2k}=0$ and $c_{2k-1} = \dfrac{2(-1)^{k+1}}{\pi(2k-1)}$ for all $k \in \mathbb{N}$; thus the Fourier series representation of f is given by

$$s(f,x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^\infty \frac{(-1)^k}{2k-1}\cos(2k-1)x\,.$$

Example 8.10. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x \qquad \text{if} \quad -\pi < x \leqslant \pi$$

and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. Then the Fourier coefficients of f are computed as follows: $c_k = 0$ for all $k \in \mathbb{N} \cup \{0\}$ since f is (more or less) an odd function, and

$$s_k = \frac{1}{\pi}\int_{-\pi}^\pi x\sin kx\,dx = \frac{2}{\pi}\int_0^\pi x\sin kx\,dx = \frac{2}{\pi}\Big(-\frac{x\cos kx}{k}\Big|_0^\pi + \int_0^\pi \frac{\cos kx}{k}\,dx\Big) = \frac{2(-1)^{k+1}}{k}\,.$$

Therefore, the Fourier series representation of f is given by

$$s(f,x) = 2\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\sin kx\,.$$

Proposition 8.11. *Let $f : [-\pi,\pi]$ be Riemann integrable and $s_n(f,x)$ be the n-th partial sum of the Fourier series representation of f. Then*

$$\int_{-\pi}^\pi \big|f(x) - s_n(f,x)\big|^2\,dx \leqslant \int_{-\pi}^\pi \big|f(x) - p(x)\big|^2\,dx \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T})\,.$$

Proof. We note that if $p \in \mathscr{P}_n(\mathbb{T})$, then $s_n(p,\cdot) = p$ and

$$\int_{-\pi}^{\pi} \big(f(x) - s_n(f,x)\big)p(x)\,dx = 0\,.$$

Therefore, if $p \in \mathscr{P}_n(\mathbb{T})$,

$$\begin{aligned}\int_{-\pi}^{\pi} \big|f(x) - p(x)\big|^2\,dx &= \int_{-\pi}^{\pi} \big|f(x) - s_n(f,x) + s_n(f,x) - p(x)\big|^2\,dx \\ &= \int_{-\pi}^{\pi} \big|f(x) - s_n(f,x)\big|^2\,dx + \int_{-\pi}^{\pi} \big|s_n(f-p,x)\big|^2\,dx \qquad (8.14)\end{aligned}$$

which concludes the proposition. □

Theorem 8.12. *Let $f \in \mathscr{C}(\mathbb{T})$. Then*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \big|s_n(f,x) - f(x)\big|^2\,dx = 0 \qquad (8.15)$$

and

$$\int_{-\pi}^{\pi} \big|f(x)\big|^2\,dx = \pi\Big[\frac{c_0^2}{2} + \sum_{k=1}^{\infty}(c_k^2 + s_k^2)\Big]\,. \qquad \textit{(Parseval's identity)} \qquad (8.16)$$

Proof. Let $\varepsilon > 0$ be given. By the denseness of the trigonometric polynomials in $\mathscr{C}(\mathbb{T})$, there exists $h \in \mathscr{P}(\mathbb{T})$ such that $\sup\limits_{x\in\mathbb{R}} \big|f(x) - h(x)\big| < \sqrt{\dfrac{\varepsilon}{2\pi}}$. Suppose that $h \in \mathscr{P}_N(\mathbb{T})$. Then by Proposition 8.11,

$$\int_{-\pi}^{\pi} \big|f(x) - s_N(f,x)\big|^2\,dx \leqslant \int_{-\pi}^{\pi} \big|f(x) - h(x)\big|^2\,dx < \varepsilon\,.$$

Since $s_N(f,\cdot) \in \mathscr{P}_n(\mathbb{T})$ if $n \geqslant N$, we must have

$$\int_{-\pi}^{\pi} \big|f(x) - s_n(f,x)\big|^2\,dx \leqslant \int_{-\pi}^{\pi} \big|f(x) - s_N(f,x)\big|^2\,dx \leqslant \varepsilon \qquad \forall\, n \geqslant N\,;$$

thus (8.15) is concluded. Finally, using (8.14) with $p = 0$ we obtain that

$$\int_{-\pi}^{\pi} \big|f(x)\big|^2\,dx = \int_{-\pi}^{\pi} \big|s_n(f,x)\big|^2\,dx + \int_{-\pi}^{\pi} \big|f(x) - s_n(f,x)\big|^2\,dx\,;$$

thus passing to the limit as $n \to \infty$ and using the fact that $\displaystyle\int_{-\pi}^{\pi} \big|s_n(f,x)\big|^2\,dx = \pi\Big[\frac{c_0^2}{2} + \sum_{k=1}^{n}(c_k^2 + s_k^2)\Big]$ we conclude (8.16). □

Remark 8.13. Identities (8.15) and (8.16) also hold for Riemann integrable function $f : [-\pi,\pi] \to \mathbb{R}$. Assuming this, then Example 8.10 provides that

$$\int_{-\pi}^{\pi} x^2\,dx = \pi \sum_{k=1}^{\infty} \frac{4}{k^2}$$

which implies that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

8.3 The Fourier Convergence Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-period function and Riemann integrable over $[-\pi, \pi]$. The n-th partial sum of the Fourier series representation of f is given by

$$\begin{aligned}
s_n(f, x) &= \frac{c_0}{2} + \sum_{k=1}^{n} \big(c_k \cos kx + s_k \sin kx\big) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx + \sum_{k=1}^{n} \Big[\Big(\frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ky\, dx\Big) \cos kx + \Big(\frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ky\, dy\Big) \sin kx\Big] \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} f(y)\big(\cos ky \cos kx + \sin ky \sin kx\big)\, dy \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y)\Big(\frac{1}{2} + \sum_{k=1}^{n} \cos k(x-y)\Big) dy\,.
\end{aligned}$$

Since $\displaystyle\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$, we conclude that

$$s_n(f, x) = \int_{-\pi}^{\pi} f(y) \frac{\sin(n+\frac{1}{2})(x-y)}{2\pi \sin\frac{x-y}{2}}\, dy\,.$$

This induces the following

Definition 8.14. The function

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\pi \sin\frac{x}{2}} \tag{8.17}$$

is called the ***Dirichlet kernel***.

Definition 8.15 (Convolutions)**.** For 2π-period functions f, g, the convolution of f and g, denoted by $f \star g$, is the function

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(x-y) g(y)\, dy\,.$$

Using this definition, we have $s_n(f, x) = (D_n \star f)(x)$. We note that similar to (a) of Proposition 7.38, by the periodicity of f and g we also have $f \star g = g \star f$.

Theorem 8.16. *For any $f \in \mathscr{C}^1(\mathbb{T})$; that is, f is 2π-periodic continuously differentiable function, $s_n(f, \cdot) = D_n \star f$ converges to f uniformly as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be given. Define $\delta = \dfrac{\varepsilon}{4(\|f'\|_\infty + 1)}$, where $\|\cdot\|_\infty$ denotes the maximum of a function. Since $\dfrac{1}{n+\frac{1}{2}} \to 0$ as $n \to \infty$ and f, f' are bounded, there exists $N > 0$ such that

$$\frac{1}{2\pi}\Big[\frac{2\|f\|_\infty}{(n+\frac{1}{2})\sin\frac{\delta}{2}} + \frac{\pi\|f'\|_\infty}{(n+\frac{1}{2})\sin\frac{\delta}{2}} + \frac{\pi\|f\|_\infty}{(n+\frac{1}{2})\sin^2\frac{\delta}{2}}\Big] < \frac{\varepsilon}{4} \quad \text{whenever } n \geqslant N\,.$$

Since $\displaystyle\int_{\mathbb{T}} D_n(x-y)\, dy = 1$ for all $x \in \mathbb{T}$,

$$\begin{aligned}
s_n(f, x) - f(x) = (D_n \star f - f)(x) &= \int_{-\pi}^{\pi} D_n(x-y)\big(f(y) - f(x)\big) dy \\
&= \int_{-\pi}^{\pi} D_n(y)\big(f(x-y) - f(x)\big) dy\,.
\end{aligned}$$

We break the integral into two parts: one is the integral over $|y| \leqslant \delta$ and the other is the integral over $\delta < |y| \leqslant \pi$. Since $f \in \mathscr{C}^1(\mathbb{T})$,

$$|f(x-y) - f(x)| \leqslant \sup_{x\in\mathbb{R}} |f(x)||y| \equiv \|f'\|_\infty |y|\,;$$

thus by the fact that $\dfrac{2}{\pi}|x| \leqslant \sin|x|$ for $|x| \leqslant \pi$,

$$\begin{aligned}
&\Big|\int_{|y|\leqslant\delta} D_n(y)\big(f(x-y) - f(x)\big)dy\Big| \\
&\qquad \leqslant \frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{\big|f(x-y)-f(x)\big|}{\big|\sin\frac{y}{2}\big|}dy \leqslant \frac{\|f'\|_\infty}{2\pi}\int_{-\delta}^{\delta}\frac{y}{\sin\frac{y}{2}}dy \leqslant \|f'\|_\infty\delta < \frac{\varepsilon}{4}\,. \qquad (8.18)
\end{aligned}$$

As for the integral over $\delta < |y| \leqslant \pi$, we have

$$\begin{aligned}
&\int_\delta^\pi \sin\big(n+\frac{1}{2}\big)y\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}dy \\
&\quad = -\frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}\Big|_{y=\delta}^{y=\pi} + \int_\delta^\pi \frac{\cos\big(n+\frac{1}{2}\big)y}{n+\frac{1}{2}}\frac{d}{dy}\frac{f(x-y)-f(x)}{\sin\frac{y}{2}}dy \\
&\quad = \frac{\cos\big(n+\frac{1}{2}\big)\delta}{n+\frac{1}{2}}\frac{f(x+\delta)-f(x)}{\sin\frac{\delta}{2}} - \int_\delta^\pi \frac{\cos(n+\frac{1}{2})y}{n+\frac{1}{2}}\frac{f'(x+y)}{\sin\frac{y}{2}}\,dy \\
&\qquad - \int_\delta^\pi \frac{\cos(n+\frac{1}{2})y}{n+\frac{1}{2}}\frac{\cos\frac{y}{2}\big(f(x-y)-f(x)\big)}{2\sin^2\frac{y}{2}}\,dy\,.
\end{aligned}$$

Therefore, if $n \geqslant N$,

$$\begin{aligned}
&\Big|\int_\delta^\pi \sin\big(n+\frac{1}{2}\big)y\frac{f(x-y)-f(x)}{2\pi\sin\frac{y}{2}}dy\Big| \\
&\qquad \leqslant \frac{1}{2\pi}\Big[\frac{2\|f\|_\infty}{(n+\frac{1}{2})\sin\frac{\delta}{2}} + \frac{\pi\|f'\|_\infty}{(n+\frac{1}{2})\sin\frac{\delta}{2}} + \frac{\pi\|f\|_\infty}{(n+\frac{1}{2})\sin^2\frac{\delta}{2}}\Big] < \frac{\varepsilon}{4}\,. \qquad (8.19)
\end{aligned}$$

Similarly, if $n \geqslant N$,

$$\Big|\int_{-\pi}^{-\delta} \sin\big(n+\frac{1}{2}\big)y\frac{f(x-y)-f(x)}{2\pi\sin\frac{y}{2}}dy\Big| < \frac{\varepsilon}{4}\,. \qquad (8.20)$$

Therefore, the combination of (8.18)-(8.20) implies that

$$\sup_{x\in\mathbb{R}}\Big|\int_{-\pi}^{\pi}\sin(n+\frac{1}{2})y\frac{f(x+y)-f(x)}{\sin\frac{y}{2}}dy\Big| < \varepsilon \quad \text{whenever } n \geqslant N\,.$$

This implies that $\lim\limits_{n\to\infty}\sup\limits_{x\in\mathbb{R}}\big|(D_n \star f)(x) - f(x)\big| = 0$ or $D_n \star f$ converges uniformly to f. ▫

Remark 8.17. Given a continuous function g with period $2L$, let $f(x) = g\big(\frac{Lx}{\pi}\big)$. Then f is a continuous function with period 2π, and the Fourier series representation of f is given by

$$s(f,x) = \frac{c_0}{2} + \sum_{k=1}^{n}(c_k\cos kx + s_k\sin kx)\,,$$

where c_k and s_k are given by (8.13). Now, define the Fourier series representation of g by $s(g,x) = s\big(f,\frac{\pi x}{L}\big)$. Then the Fourier series representation of g is given by

$$s(g,x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} \Big(c_k \cos\frac{k\pi x}{L} + s_k \sin\frac{k\pi x}{L}\Big)\,,$$

where $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ is also called the Fourier coefficients associated with g and are given by

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} g\big(\frac{Lx}{\pi}\big)\cos kx\,dx = \frac{1}{L}\int_{-L}^{L} g(x)\cos\frac{k\pi x}{L}\,dx$$

and similarly, $s_k = \dfrac{1}{L}\displaystyle\int_{-L}^{L} g(x)\sin\frac{k\pi x}{L}\,dx$. Moreover, the change of variable formula implies that

$$\int_{-L}^{L} \big|g(x)\big|^2\,dx = \int_{-L}^{L}\Big|f\big(\frac{\pi x}{L}\big)\Big|^2\,dx = \frac{L}{\pi}\int_{-\pi}^{\pi}\big|f(x)\big|^2\,dx = L\Big[\frac{c_0^2}{2} + \sum_{k=1}^{\infty}(c_k^2 + s_k^2)\Big]\,. \tag{8.21}$$

Identity (8.21) is the Parseval identity for $2L$-periodic function g.

Definition 8.18. A function f is said to be ***piecewise continuous*** on an interval $[\alpha, \beta]$ if the interval can be partitioned by a finite number of points $\alpha = x_0 < x_1 < \cdots < x_n = \beta$ such that

1. f is continuous on each open sub-interval (x_{i-1}, x_i).
2. f approaches a finite limit as the end-points of each sub-interval are approached from within the sub-interval.

Theorem 8.19. *Suppose that f and f' are piecewise continuous on the interval $[-L, L)$ and $f(x + L) = f(x)$ for all $x \in \mathbb{R}$. Then*

$$\frac{f(x^+) + f(x^-)}{2} = s(f,x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}\big(c_k\cos kx + s_k \sin kx\big)\,,$$

where $f(x^{\pm}) = \lim\limits_{y\to x^{\pm}} f(x)$ denotes the one-sided limit of f at x.

Example 8.20. Let $f(x) = \begin{cases} 0 & \text{if } -L < x < 0\,, \\ L & \text{if } 0 < x < L\,, \end{cases}$ and $f(x + 2L) = f(x)$ for all $x \in \mathbb{R}$. Then

$$s(f,x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\frac{(2k-1)\pi x}{L}\,.$$

The following figure demonstrates the graph of f and the 8-th partial sum of f

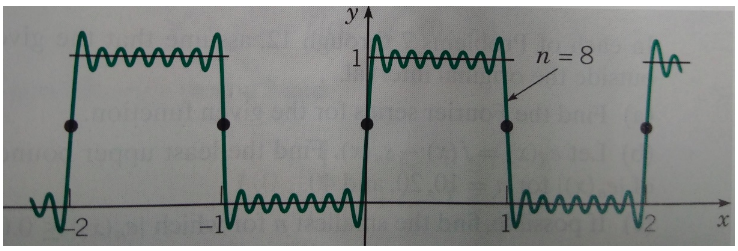


Figure 6: The partial sum $s_8(f,x) = \dfrac{L}{2} + \dfrac{2L}{\pi}\displaystyle\sum_{k=1}^{8}\frac{1}{2k-1}\sin\frac{(2k-1)\pi x}{L}\,.$

One can use the following matlab® code to generate the picture above. The reader can change the value of N and see how the Fourier series converges to the step function.

```
N = 8; % degree of trigonometric polynomial
L = 1; % half of the period
x = -2.2:0.01:2.2;

% Computing the N-th partial sum of the Fourier series
S = L/2*ones(1,length(x));
for k = 1:N
  S = S + 2*L*sin((2*k-1)*pi*x/L)/(pi*(2*k-1));
end

% Plot the N-th partial sum of the Fourier series
plot(x,S);
hold on;

% Plot the step function
t = -2.5:0.01:2.5;
s = (t < -1).*(t > -2).*ones(1,length(t))...
  + (t < 1).*(t > 0).*ones(1,length(t)) + (t > 2).*ones(1,length(t));
plot(t,s);
```

Theorem 8.21 (Gibbs' Phenomena). *Let f and f' be piecewise continuous on the interval $[-L, L)$ and $f(x+L) = f(x)$ for all $x \in \mathbb{R}$. Suppose that at some point x_0 the limit from the left $f(x_0^-)$ and the limit from the right $f(x_0^+)$ of the function f exist and differ by a non-zero gap a:*

$$f(x_0^+) - f(x_0^-) = a \neq 0,$$

then there exists a constant $c > 0$, independent of f, x_0 and L (in fact, $c = \frac{1}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx - \frac{1}{2} \approx 0.089490$), such that

$$\lim_{n\to\infty} s_n\big(f, x_0 + \frac{L}{2n}\big) = f(x_0^+) + ca\,, \tag{8.22a}$$

$$\lim_{n\to\infty} s_n\big(f, x_0 - \frac{L}{2n}\big) = f(x_0^-) - ca\,. \tag{8.22b}$$

8.4 Even and Odd Functions

In this section, we consider the Fourier series representation of $2L$-periodic even or odd functions. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called an even (resp. odd) function if $f(-x) = f(x)$ (resp. $f(-x) = -f(x)$) for all $x \in \mathbb{R}$, and we note that if $f : \mathbb{R} \to \mathbb{R}$ is an even (resp. odd) function,

$$\int_{-M}^{M} f(x)\,dx = 2\int_0^M f(x)\,dx \qquad \Big(\text{resp. } \int_{-M}^{M} f(x)\,dx = 0\Big) \qquad \forall\, M > 0\,.$$

Therefore, if f is a $2L$-periodic even function, the Fourier series representation of f is given by

$$s(f,x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\frac{k\pi x}{L}\,, \tag{8.23}$$

while if f is a $2L$-periodic odd function, the Fourier series representation of f is given by

$$s(f,x) = \sum_{k=1}^{\infty} s_k \sin\frac{k\pi x}{L}\,, \tag{8.24}$$

where $\{c_k\}_{k=0}^{\infty}$, $\{s_k\}_{k=1}^{\infty}$ are given by

$$c_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx\,, \qquad s_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx\,. \tag{8.25}$$

Definition 8.22. Let $f:[0,L]\to\mathbb{R}$ be a function. Then ***even*** (***resp. odd***) ***extension*** of f is the function $f_e:[-L,L]\to\mathbb{R}$ (resp. $f_o:[-L,L]\to\mathbb{R}$) such that

$$f_e(x) = \begin{cases} f(x) & \text{if } x\in[0,L]\,, \\ f(-x) & \text{if } x\in[-L,0)\,. \end{cases} \qquad \Big(\text{resp. } f_o(x) = \begin{cases} f(x) & \text{if } x\in[0,L]\,, \\ -f(-x) & \text{if } x\in[-L,0)\,. \end{cases}\Big)$$

The ***even*** (***resp. odd***) ***periodic extension*** of f is a $2L$-periodic function which coincides with f_e (resp. f_o) in the interval $[-L,L]$.

Definition 8.23. Let $f:[0,L]\to\mathbb{R}$ be a function. The Fourier series representation of the even (resp. odd) extension of f is called the ***Fourier cosine*** (***resp. sine***) ***series*** of f.

Using (8.23)-(8.25), the Fourier cosine and sine series of $f:[0,L]\to\mathbb{R}$ is

$$s(f_e,x) = \frac{1}{L}\int_0^L f(y)dy + \frac{2}{L}\sum_{k=1}^{\infty}\Big(\int_0^L f(y)\cos\frac{k\pi y}{L}dy\Big)\cos\frac{k\pi x}{L}$$

and

$$s(f_o,x) = \frac{2}{L}\sum_{k=1}^{\infty}\Big(\int_0^L f(y)\sin\frac{k\pi y}{L}dy\Big)\sin\frac{k\pi x}{L}\,,$$

respectively. By Theorem 8.19, if $f:[0,L]\to\mathbb{R}$ is piecewise continuous with piecewise continuous f', then

$$\frac{f(x^+)+f(x^-)}{2} = s(f_e,x) = s(f_o,x) \qquad \forall\, x\in[0,L]\,.$$

Moreover, Remark 8.13 and (8.21) imply that

$$\begin{aligned}\int_0^L |f(x)|^2\,dx &= \frac{1}{2}\int_{-L}^{L}|f_e(x)|^2\,dx = \frac{L}{2}\Big[\frac{2}{L^2}\Big(\int_0^L f(y)\,dy\Big)^2 + \frac{4}{L^2}\sum_{k=1}^{\infty}\Big(\int_0^L f(y)\cos\frac{k\pi y}{L}dy\Big)^2\Big] \\ &= \frac{1}{L}\Big(\int_0^L f(y)\,dy\Big)^2 + \frac{2}{L}\sum_{k=1}^{\infty}\Big(\int_0^L f(y)\cos\frac{k\pi y}{L}dy\Big)^2\Big]\end{aligned}$$

and similarly,

$$\int_0^L |f(x)|^2\,dx = \frac{1}{2}\int_{-L}^{L}|f_o(x)|^2\,dx = \frac{2}{L}\sum_{k=1}^{\infty}\Big(\int_0^L f(y)\sin\frac{k\pi y}{L}dy\Big)^2\,.$$

Example 8.24. By Example 8.8 and 8.10, we conclude that

$$x = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} \qquad \forall\, x \in [0, \pi)\,.$$

The Parseval identity provides that

$$\frac{2\pi^3}{3} = \pi \sum_{k=1}^{\infty} \frac{4}{k^2} = \pi\Big[\frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}\Big]\,.$$

8.5 Separation of Variables; Heat Conduction in a Rod

Consider the heat distribution on a rod of length L: Parameterize the rod by $[0, L]$, and let t be the time variable. Let $\rho(x)$, $s(x)$, $\kappa(x)$ denote the density, specific heat, and the thermal conductivity of the rod at position $x \in (0, L)$, respectively, and $u(x, t)$ denote the temperature at position x and time t. For $0 < x < L$, and $\Delta x, \Delta t \ll 1$,

$$\int_x^{x+\Delta x} \rho(y)s(y)\big[u(y, t+\Delta t) - u(y, t)\big]\, dy = \int_t^{t+\Delta t} \big[-\kappa(x)u_x(x, t') + \kappa(x+\Delta x)u_x(x+\Delta x, t')\big]\, dt'\,,$$

where the left-hand side denotes the change of the total heat in the small section $(x, x+\Delta x)$, and the right-hand side denotes the heat flows from outside. Divide both sides by $\Delta x \Delta t$ and letting Δx and Δt approach zero, if all the functions appearing in the equation above are smooth enough, we find that

$$\rho(x)s(x)u_t(x, t) = \big[\kappa(x)u_x(x, t)\big]_x \qquad 0 < x < L\,, \quad t > 0\,. \tag{8.26}$$

Assuming uniform rod; that is, ρ, s, κ are constant, then (8.26) reduces to that

$$u_t(x, t) = \alpha^2 u_{xx}(x, t)\,, \qquad 0 < x < L\,, \quad t > 0\,, \tag{8.27a}$$

where $\alpha^2 = \dfrac{\kappa}{\rho s}$ is called the ***thermal diffusivity***.

To determine the state of the temperature, we need to impose that initial condition

$$u(x, 0) = f(x) \qquad 0 < x < L \tag{8.27b}$$

and a boundary condition. In this section, we consider the Dirichlet boundary condition

$$u(0, t) = u(L, t) = 0 \qquad t > 0\,. \tag{8.27c}$$

Method of Separation of Variables: Assume that the solution u is a product of two functions, one depending only on x and the other depending only on t; thus

$$u(x, t) = X(x)T(t)\,.$$

Then (8.27a) implies that

$$T'(t)X(x) = \alpha^2 T(t)X''(x) \qquad 0 < x < L\,, \quad t > 0\,.$$

Rearranging terms, we obtain that

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}\,.$$

Since the left-hand side is a function of x and the right-hand side is a function of t, we must have

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -\lambda\,.$$

for some constant λ. In other words, X and T satisfy

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \qquad 0 < x < L\,, \\ T'(t) + \alpha^2\lambda T(t) &= 0 \qquad t > 0\,. \end{aligned}$$

Since $u(0,t) = u(L,t) = 0$ for all $t > 0$, for $X(x)T(t)$ to be a solution, we must have $X(0) = X(L) = 0$. As discussed before, in order to have non-trivial solution, λ has to be positive and using (8.7) we find that

$$\lambda_k = \frac{k^2\pi^2}{L^2} \quad \text{and} \quad X_k(x) = \sin\frac{k\pi x}{L}\,.$$

This in turn implies that

$$T'(t) + \frac{k^2\pi^2\alpha^2}{L^2}T(t) = 0\,;$$

thus $T(t) = e^{-\frac{k^2\pi^2\alpha^2}{L^2}t}$. As a consequence, we conclude that

$$u_k(x,t) = e^{-\frac{k^2\pi^2\alpha^2}{L^2}t}\sin\frac{k\pi x}{L}\,.$$

This u_k satisfies (8.27a) and (8.27c) for all $k \in \mathbb{N}$. By the superposition principle, we also expect that the linear combination of u_k's satisfies (8.27a) and (8.27c).

To satisfy the initial condition $u(x,0) = f(x)$, we first find the Fourier sine series of f and find that

$$f(x) = \sum_{k=1}^{\infty} s_k \sin\frac{k\pi x}{L}\,, \qquad s_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx\,.$$

Define

$$u(x,t) = \sum_{k=1}^{\infty} s_k e^{-\frac{k^2\pi^2\alpha^2}{L^2}t}\sin\frac{k\pi x}{L}\,, \qquad s_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx\,.$$

Then if the differentiation in both x and t commutes with the infinite sum, then u given above solves (8.27).

8.6 Other Heat Conduction Problems

8.6.1 Non-homeogeneous Dirichlet boundary conditions

In this sub-section we consider the heat equation (8.27a,b) with non-homogeneous Dirichlet boundary condition

$$u(0,t) = T_1 \quad \text{and} \quad u(L,t) = T_2 \qquad t > 0\,. \tag{8.27d}$$

Define $v(x) = (T_2 - T_1)\dfrac{x}{L} + T_1$. Then $v(0) = T_1$ and $v(L) = T_2$. Letting $w(x,t) = u(x,t) - v(x)$, we find that w satisfies

$$\begin{aligned} w_t(x,t) &= \alpha^2 w_{xx}(x,t) && 0 < x < L\,, \quad t > 0\,, \\ w(x,0) &= f(x) - v(x) && 0 < x < L\,, \\ w(0,t) &= w(L,t) = 0 && t > 0\,. \end{aligned}$$

By the discussion in the previous section, we find that

$$w(x,t) = \sum_{k=1}^{\infty} b_k e^{-\frac{k^2\pi^2\alpha^2}{L^2}t} \sin\frac{k\pi x}{L}\,, \qquad b_k = \frac{2}{L}\int_0^L (f(x) - v(x)) \sin\frac{k\pi x}{L}\,dx\,.$$

Therefore, the solution to (8.27a,b,d) is given by

$$\begin{aligned} u(x,t) &= v(x) + w(x,t) \\ &= (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{k=1}^{\infty} b_k e^{-\frac{k^2\pi^2\alpha^2}{L^2}t} \sin\frac{k\pi x}{L}\,, \end{aligned}$$

where

$$b_k = \frac{2}{L}\int_0^L \Big(f(x) - (T_2 - T_1)\frac{x}{L} - T_1\Big) \sin\frac{k\pi x}{L}\,dx\,.$$

Since the temperature at the ends of the rod are fixed to be some constants, we expect that $u(x,t) \to v(x)$ as $t \to \infty$. To see this mathematically, we consider the case that $t \gg 1$. By the fact that $|b_k| \leqslant \dfrac{2}{L}\displaystyle\int_0^L \big|f(x) - v(x)\big|\,dx$, we have

$$\begin{aligned} \max_{x\in[0,L]} \big|u(x,t) - v(x)\big| &\leqslant \sum_{k=1}^{\infty} |b_k| e^{-\frac{k^2\pi^2\alpha^2}{L^2}t} \leqslant \Big(\frac{2}{L}\int_0^L \big|f(x) - v(x)\big|\,dx\Big) \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2\alpha^2}{L^2}t} \\ &\leqslant e^{-\frac{\pi^2\alpha^2}{L^2}(t-1)} \Big(\frac{2}{L}\int_0^L \big|f(x) - v(x)\big|\,dx\Big) \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2\alpha^2}{L^2}}\,. \end{aligned}$$

Since $\displaystyle\sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2\alpha^2}{L^2}} < \infty$, we conclude that

$$\lim_{t\to\infty} \max_{x\in[0,L]} \big|u(x,t) - v(x)\big| = 0\,.$$

This shows that $u(\cdot,t) \to v$ uniformly on $[0,L]$ as $t \to \infty$, and this further shows that

$$\lim_{t\to\infty} \int_0^L \big|u(x,t) - v(x)\big|^2\,dx = 0\,.$$

On the other hand, for each fixed $t > 0$, we can treat

$$\sum_{k=1}^{\infty} b_k e^{-\frac{k^2\pi^2\alpha^2}{L^2}t} \sin\frac{k\pi x}{L}$$

as the Fourier sine series of $u(x,t) - v(x)$; thus the Parseval identity implies that

$$\begin{aligned} \int_0^L \big|u(x,t) - v(x)\big|^2\,dx &= L\sum_{k=1}^{\infty} b_k^2 e^{-\frac{2k^2\pi^2\alpha^2}{L^2}t} \leqslant e^{-\frac{2\pi^2\alpha^2}{L^2}t} \sum_{k=1}^{\infty} b_k^2 \\ &= Le^{-\frac{2\pi^2\alpha^2}{L^2}t} \int_0^L |f(x)|^2\,dx \to 0 \qquad \text{as} \quad t \to \infty. \end{aligned}$$

8.6.2 Homogeneous Neumann boundary conditions

In this sub-section we consider the heat equation (8.27a,b) with non-homogeneous Dirichlet boundary condition

$$u_x(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0 \qquad t > 0\,. \tag{8.27e}$$

We remark that this boundary condition means the end of the rod are insulated.

Now we apply the method of separation of variables. Suppose $u(x,t) = X(x)T(t)$ is a solution to (8.27a) and (8.27e). Then again

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -\lambda\,.$$

for some constant λ, or equivalently, X and T satisfy

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \qquad 0 < x < L\,,\\ T'(t) + \alpha^2\lambda T(t) &= 0 \qquad t > 0\,. \end{aligned}$$

Since $u_x(0,t) = u_x(L,t) = 0$ for all $t > 0$, for $X(x)T(t)$ satisfying (8.27e), we must have $X'(0) = X'(L) = 0$. As discussed before, in order to have non-trivial solution, λ has to be non-negative and using (8.7) we find that

$$\lambda_k = \frac{k^2\pi^2}{L^2} \quad \text{and} \quad X_k(x) = \cos\frac{k\pi x}{L} \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$

This in turn implies that

$$T'(t) + \frac{k^2\pi^2\alpha^2}{L^2}T(t) = 0\,;$$

thus $T(t) = e^{-\frac{k^2\pi^2\alpha^2}{L^2}t}$. As a consequence, we conclude that

$$u_k(x,t) = e^{-\frac{k^2\pi^2\alpha^2}{L^2}t}\cos\frac{k\pi x}{L}\,.$$

This u_k satisfies (8.27a) and (8.27e) for all $k \in \mathbb{N} \cup \{0\}$.

To satisfy the initial condition $u(x,0) = f(x)$, we first find the Fourier cosine series of f and find that

$$f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\frac{k\pi x}{L}\,, \qquad c_k = \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx\,.$$

Define

$$u(x,t) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k e^{-\frac{k^2\pi^2\alpha^2}{L^2}t}\cos\frac{k\pi x}{L}\,, \qquad c_k = \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx\,.$$

Then if the differentiation in both x and t commutes with the infinite sum, then u given above solves (8.27a,b,e).

Since the ends of the rod are insulated, we expect that the temperature converges to the average temperature $\frac{1}{L}\int_0^L f(x)\,dx$. To see this, we note that $\frac{c_0}{2}$ is the average temperature, and as in the previous case we have

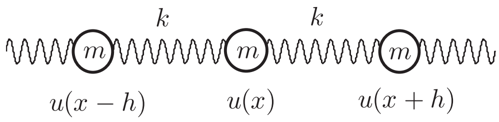
$$\max_{x\in[0,L]}\Big|u(x,t) - \frac{c_0}{2}\Big| \leqslant e^{-\frac{\pi^2\alpha^2}{L^2}(t-1)}\Big(\frac{2}{L}\int_0^L \big|f(x)\big|\,dx\Big)\sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2\alpha^2}{L^2}} \to 0 \quad \text{as} \quad t \to \infty\,.$$

8.7 The Wave Equations

In this section we consider the wave equations $u_{tt} = c^2 u_{xx}$.

8.7.1 Models

1. From Hooke's law: imagine an array of little weights of mass m interconnected with massless springs of length h, and the springs have a stiffness of k (see the figure).



If $u(x)$ measures the distance from the equilibrium of the mass situated at x, then the forces exerted on the mass m at the location x are

$$\begin{aligned} F_{\text{Newton}} &= ma = m\frac{\partial^2 u}{\partial t^2}(x,t) \\ F_{\text{Hooke}} &= k[u(x+h,t) - u(x,t)] - k[u(x,t) - u(x-h,t)] \\ &= k[u(x+h,t) - 2u(x,t) + u(x-h,t)] \,. \end{aligned}$$

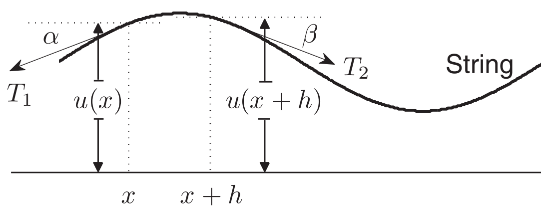
If the array of weights consists of N weights spaced evenly over the length $L = Nh$ of total mass $M = Nm$, and the total stiffness of the array $K = k/N$, then

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{KL^2}{M}\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} \,.$$

Taking the limit $N \to \infty$, $h \to 0$ (and assuming smoothness) we obtain

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \,. \tag{8.28a}$$

(2) Equation of vibrating string: let $u(x,t)$ measure the distance of a string from its equilibrium.



Assuming only motion in the vertical direction, the horizontal component of tensions T_1 and T_2 have to be the same:

$$T_1 \cos\alpha = T_2 \cos\beta \approx T \,.$$

The difference of the vertical component of T_1 and T_2 induces the motion in the vertical direction:

$$\begin{aligned} m\frac{\partial^2 u}{\partial t^2}(x,t) &\approx T_2 \sin\beta - T_1 \sin\alpha = (T_2 \cos\beta)\tan\beta - (T_1 \cos\alpha)\tan\alpha \\ &\approx [T(x+h)u_x(x+h,t) - T(x)u_x(x,t)] \,. \end{aligned}$$

If ρ is the density of the string, then $m = \rho h$; hence

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) \approx \frac{T(x+h)u_x(x+h,t) - T(x)u_x(x,t)}{h}.$$

Taking the limit $h \to 0$, we obtain

$$\rho u_{tt}(x,t) = \big[T(x)u_x(x,t)\big]_x. \tag{8.29}$$

If T is constant, then (8.29) reduces to

$$u_{tt}(x,t) = c^2 u_{xx}(x,t). \tag{8.28)a$$

To determine the state of the displacement u, we need to impose that initial condition

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \qquad 0 < x < L \tag{8.27b}$$

and a boundary condition. In this section, we consider the Dirichlet boundary condition

$$u(0,t) = u(L,t) = 0 \qquad t > 0. \tag{8.28c}$$

Again, applying the method of separation of variables, we assume that $u(x,t) = X(x)T(t)$ satisfying (8.28a,c) and find that

$$\begin{aligned} T''(t)X(x) &= c^2 T(t)X''(x) \qquad 0 < x < L, \quad t > 0, \\ X(0)T(t) &= X(L)T(t) = 0 \qquad t > 0. \end{aligned}$$

Therefore,

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \qquad 0 < x < L, \quad t > 0$$

for some constant λ. Taking the boundary condition $X(0) = X(L) = 0$ into account, we find that

$$\lambda = \lambda_k = \frac{k^2\pi^2}{L^2}, \quad X(x) = X_k(x) = \sin\frac{k\pi x}{L},$$

and

$$T(t) = T_k(t) = c_k \cos\frac{k\pi ct}{L} + s_k \sin\frac{k\pi ct}{L},$$

in which $k \in \mathbb{N}$. Define $u_k(x,t) = X_k(x)T_k(t)$, then we look for c_k, s_k such that the "formal" solution of (8.28a,b,c) can be expressed by

$$u(x,t) = \sum_{k=1}^{\infty} \Big(c_k \cos\frac{k\pi ct}{L} + s_k \sin\frac{k\pi ct}{L}\Big) \sin\frac{k\pi x}{L}.$$

To satisfy the initial condition (8.28b), we have

$$c_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx \quad \text{and} \quad s_k = \frac{2}{k\pi c}\int_0^L g(x)\sin\frac{k\pi x}{L}\,dx.$$

8.7.2 The d'Alembert formula

Let $v(x,t) = \sum\limits_{k=1}^{\infty} c_k \cos \frac{k\pi ct}{L} \sin \frac{k\pi x}{L}$ and $w(x,t) = \sum\limits_{k=1}^{\infty} s_k \sin \frac{k\pi ct}{L} \sin \frac{k\pi x}{L}$. Then

$$v(x,t) = \sum_{k=1}^{\infty} \frac{c_k}{2} \Big(\sin \frac{k\pi(x+ct)}{L} + \sin \frac{k\pi(x-ct)}{L} \Big) = \frac{F(x+ct) + F(x-ct)}{2}$$

and

$$w(x,t) = \sum_{k=1}^{\infty} \frac{s_k}{2} \Big(\cos \frac{k\pi(x-ct)}{L} - \cos \frac{k\pi(x+ct)}{L} \Big) = \frac{H(x-ct) - H(x+ct)}{2},$$

where $F(x) = \sum\limits_{k=1}^{\infty} c_k \sin \frac{k\pi x}{L}$ and $H(x) = \sum\limits_{k=1}^{\infty} s_k \cos \frac{k\pi x}{L}$. We note that F is the odd period extension of f while

$$H'(x) = -\sum_{k=1}^{\infty} \frac{k\pi}{L} c_k \cos \frac{k\pi x}{L}$$

is the odd period extension of $-\frac{1}{c} g$. Let G be the odd periodic extension of g. Then

$$\frac{H(x-ct) - H(x+ct)}{2} = \frac{1}{2c} \int_{x-ct}^{x+ct} G(\xi)\, d\xi\,;$$

thus the formal solution to (8.28) is given by

$$u(x,t) = \frac{F(x+ct) + F(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} G(\xi)\, d\xi\,. \tag{8.30}$$

This is called the ***d'Alembert formula***.

If F is twice differentiable, G is differentiable, then u given by (8.30) satisfies

$$\begin{aligned}
u_t(x,t) &= \frac{c}{2}\big[F'(x+ct) - F'(x-ct)\big] + \frac{1}{2}\big[H(x+ct) + H(x-ct)\big]\,,\\
u_{tt}(x,t) &= \frac{c^2}{2}\big[F''(x+ct) + F''(x+ct)\big] + \frac{c}{2}\big[H'(x+ct) - H'(x-ct)\big]\,,\\
u_x(x,t) &= \frac{1}{2}\big[F'(x+ct) + F'(x-ct)\big] + \frac{1}{2c}\big[H(x+ct) - H(x-ct)\big]\,,\\
u_{tt}(x,t) &= \frac{1}{2}\big[F''(x+ct) + F''(x+ct)\big] + \frac{1}{2c}\big[H'(x+ct) - H'(x-ct)\big]\,.
\end{aligned}$$

Therefore, $u_{tt}(x,t) = c^2 u_{xx}(x,t)$ for $0 < x < L$ and $t > 0$. Moreover, u clearly satisfies (8.28b) and (8.28c); thus u given by (8.30)) solves (8.28).