# Numerical Analysis I MA3021 Midterm

National Central University, May. 06, 2020

**Problem 1.** Let $f$ be a given real-valued function in $C([a,b])$, and $x_0,\ x_1,\ \cdots,\ x_n$ be $n+1$ distinct numbers in $[a,b]$.

1. (8%) What is the $n$-th Lagrange interpolating polynomial?
2. (8%) Suppose in addition that $f$ is differentiable on $[a,b]$. What is the $(2n+1)$-th Hermite interpolating polynomial?

**Problem 2.** (15%) Let $f:(a,b)\to\mathbb{R}$ be a twice differentiable function such that $\big|f'(x)\big|\geqslant K$ and $\big|f''(x)\big|\leqslant M$ for all $x\in(a,b)$, where $K,M$ are positive real numbers. Suppose that $r\in(a,b)$ is a root of $f$, and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$ for all $n\geqslant 1$ is a subset of $(a,b)$. Show that

$$|x_{n+1}-r|\leqslant\frac{M}{2K}|x_n-r|^2\qquad\forall\, n\geqslant 1\,.$$

**Problem 3.** (15%) Let $f$ be a given real-valued function in $C^{n+1}([a,b])$, and $x_0,\ x_1,\ \cdots,\ x_n$ be $n+1$ distinct numbers in $[a,b]$. Show that for each $x$ in $[a,b]$, there exists $\xi(x)\in(a,b)$ such that

$$f(x)=p(x)+\frac{1}{(n+1)!}f^{(n+1)}(\xi(x))\prod_{i=0}^{n}(x-x_i),$$

where $p(x)$ is $n$-th Lagrange interpolating polynomial for the data $(x_0,f(x_0))$, $(x_1,f(x_1))$, $\cdots$, $(x_n,f(x_n))$.

**Problem 4.** (15%) Show that if $f$ is five-times continuously differentiable in the region of interest, then

$$f'(x_0)=\frac{1}{12h}\Big[-25f(x_0)+48f(x_0+h)-36f(x_0+2h)+16f(x_0+3h)-3f(x_0+4h)\Big]+\mathcal{O}(h^4)\,.$$

**Problem 5.** Let $p_n(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ be the Legendre polynomial of degree $n$. Assume that you know that $\displaystyle\int_{-1}^{1}p(x)p_n(x)\,dx=0$ if degree$(p)<n$.

(1) (15%) Let $x_0,x_1,x_2,\cdots,x_n$ be distinct roots of the Legendre polynomial $p_{n+1}$, and $L_{n,i}$ be the Lagrange polynomial of degree $n$ satisfying $L_{n,i}(x_j)=\delta_{ij}$. If $p$ is a polynomial and degree$(p)<2(n+1)$, then

$$\int_{-1}^{1}p(x)dx=\sum_{i=0}^{n}c_ip(x_i),$$

where $c_i=\displaystyle\int_{-1}^{1}L_{n,i}(x)\,dx$.

(2) (9%) Show that $\sum\limits_{i=0}^{n}c_i=2$.

**Problem 6.** (15%) Evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using Simpson's rule so that the difference between the exact value and the approximation you obtained is less than $10^{-8}$. Explain the detail of your procedure (without exactly evaluating the integral).