Numerical Analysis I MA3021 Midterm

National Central University, Jun. 29, 2020 (due. Jul. 01, 2020 4pm)

Problem 1. Consider solving $A\boldsymbol{x} = \boldsymbol{b}$, where A is a given $n \times n$ real matrix, and $\boldsymbol{b}$ is a given n $\times$ 1 real vector, using the iterative scheme

$$\boldsymbol{x}^{(k+1)} = g\big(\boldsymbol{x}^{(k)}\big) \equiv \boldsymbol{x}^{(k)} + \omega\big(\boldsymbol{b} - A\boldsymbol{x}^{(k)}\big)\,, \tag{$\star$}$$

where $\omega \neq 0$ is a real constant.

(1) (10%) Show that $\boldsymbol{x}$ satisfies $A\boldsymbol{x} = \boldsymbol{b}$ if and only if $\boldsymbol{x}$ is a fixed-point of g.

(2) (10%) Show that if $\|\mathrm{I}_{n\times n} - \omega A\| < 1$ for some sub-ordinate matrix norm, then the iterative scheme ($\star$) converges; that is, ($\star$) produces convergent sequence $\big\{\boldsymbol{x}^{(k)}\big\}_{k=1}^{\infty}$ for any given initial guess $\boldsymbol{x}^{(0)}$.

(3) (20%) Suppose that A is a symmetric matrix with eigenvalue $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. Show that the iterative scheme ($\star$) converges if $|1 - \omega\lambda_j| < 1$ for all $1 \leqslant j \leqslant n$.

(4) (20%) Suppose that A is a symmetric positive definite matrix, and $0 < \omega \ll 1$ so that $|1-\omega\lambda| < 1$ for all eigenvalues λ of A. Define $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - A^{-1}\boldsymbol{b}$. Show that

$$\big\|\boldsymbol{e}^{(k)}\big\|_2 \leqslant \frac{|1-\omega\lambda_{\min}|^k}{\lambda_{\min}}\|\boldsymbol{b} - A\boldsymbol{x}^{(0)}\|_2 \qquad \forall\, k \in \mathbb{N}\,,$$

where $\lambda_{\min}$ is the smallest eigenvalue of A.

Problem 2. Let A be an $n \times n$ symmetric matrix with the property that the dominant eigenvalue of A is simple; that is, the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A satisfies

$$|\lambda_1| > |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n|\,.$$

Here λ_1 is called the dominant eigenvalue of A.

(1) (10%) Suppose that $\boldsymbol{v} \neq \boldsymbol{0}$ is not an eigenvector of A and $\boldsymbol{v}$ is not orthogonal to the eigenvector corresponding to the dominant eigenvalue λ. Show that

$$\lim_{m\to\infty} \frac{\boldsymbol{v}^{\mathrm{T}} A^{m+1}\boldsymbol{v}}{\boldsymbol{v}^{\mathrm{T}} A^{m}\boldsymbol{v}} = \lambda\,.$$

Hint: Write $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$ and find $\dfrac{\boldsymbol{v}^{\mathrm{T}} A^{m+1}\boldsymbol{v}}{\boldsymbol{v}^{\mathrm{T}} A^{m}\boldsymbol{v}}$.

(2) (10%) Use (1) to provide an algorithm which computes the dominant eigenvalue of a symmetric matrix (with simple dominant eigenvalue).

Problem 3. (20%) Show that

$$x_{n+1} = x_n + \frac{h}{6}\big(k_1 + 4k_2 + k_3\big)\,,$$

where

$$k_1 = f(t_n, x_n)\,, \quad k_2 = f\big(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_1\big)\,, \quad k_3 = f\big(t_n + h, x_n - hk_1 + 2hk_2\big)\,,$$

is a third order method of solving the initial value problem

$$x'(t) = f(t, x(t))\,, \qquad x(t_0) = x_0\,.$$