Exercise Problem 2

Due Nov. 23. 2022

Problem 1. Let $b, c \in \mathbb{R}$, and assume that $r^2 + br + c = 0$ has two distinct roots r_1, r_2. Solve the IVP

$$x''(t) + bx'(t) + cx(t) = f(t)\,, \qquad x(0) = x_0\,,\ x'(0) = x_1$$

by the following steps.

1. Since r_1, r_2 are two distinct roots of $r^2 + br + c = 0$, one has $b = -(r_1 + r_2)$ and $c = r_1 r_2$. Rewrite the IVP as

   $$x''(t) - (r_1 + r_2)x'(t) + r_1 r_2 x(t) = f(t) \qquad x(0) = x_0\,, x'(0) = x_1\,.$$

   Define $y(t) = x'(t) - r_1 x(t)$. Show that y satisfies

   $$y'(t) - r_2 y(t) = f(t)\,, \qquad y(0) = x_1 - r_1 x_0\,. \tag{$\star$}$$

2. Solve ($\star$) using the method of integrating factor.

3. Solve the IVP

   $$x'(t) - r_1 x(t) = y(t)\,, \qquad x(0) = x_0$$

   using the method of integrating factor again.

Hint: If you do not know how to express the solution in this general form, try to solve the IVP with the following settings: $b = -3$, $c = 2$, $f(t) = \sin t$, as well as $x_0 = x_1 = 0$.

Solution. 見新版 Lecture note（在課程網頁，有與舊版並列）的 31 頁到 32 頁。 □

Remark: From Problem 1 you should be able to see the reason that one has $\varphi_1(t) = e^{r_1 t}$ and $\varphi_2(t) = e^{r_2 t}$ as a basis of the solution space for the homogeneous case (if you are not told this fact).

Problem 2. Given one solution $\varphi_1(t) = t^2$ of the ODE

$$t^2 x''(t) - 3tx'(t) + 4x(t) = 0\,, \tag{$\star\star$}$$

solve the IVP

$$t^2 x''(t) - 3tx'(t) + 4x(t) = t^3 \ln t\,, \qquad x(1) = -2\,,\ x'(1) = -5$$

for $t > 0$. Do **NOT** use formula (2.25) in the lecture note to find another solution φ_2 to ($\star\star$) which is linearly independent of φ_1, but instead try to follow the steps of deriving (2.25) to find such φ_2.

Solution. Suppose that $\varphi_2(t) = t^2 v(t)$ is a solution to the corresponding homogeneous ODE

$$t^2 x''(t) - 3tx'(t) + 4x(t) = 0\,. \tag{$\diamond$}$$

Then

$$\begin{aligned}
&t^2\big[t^2 v(t)\big]'' - 3t\big[t^2 v(t)\big]' + 4t^2 v(t) = 0 \\
\Rightarrow\ & t^2\big[2v(t) + 4tv'(t) + t^2 v''(t)\big] - 3t\big[2tv(t) + t^2 v'(t)\big] + 4t^2 v(t) = 0 \\
\Rightarrow\ & t^4 v'' + t^3 v'(t) = 0 \Rightarrow tv'' + v'(t) = 0\,.
\end{aligned}$$

Let $y(t) = v'(t)$. Then $y' + \dfrac{1}{t} y = 0$; thus

$$\frac{d}{dt}\Big[\exp\Big(\int \frac{1}{t}\,dt\Big) y(t)\Big] = 0\,.$$

Therefore, $y(t) = \dfrac{C}{t}$ so that $v(t) = C\ln t$. This shows that $\varphi_2(t) = t^2 \ln t$ is another solution to $(\diamond)$ which is linearly independent of φ_1.

Having obtained a basis $\{\varphi_1, \varphi_2\}$ of the solution space of the corresponding homogeneous ODE, we apply formula (2.27) in the lecture to find a particular solution x_p of $(\star\star)$. First we note that the Wronskian $\mathrm{W}[\varphi_1, \varphi_2]$ is given by

$$\mathrm{W}[\varphi_1, \varphi_2](t) = \varphi_1(t)\varphi_2'(t) - \varphi_2(t)\varphi_1'(t) = t^2(2t\ln t + t) - t^2 \ln t \cdot 2t = t^3\,.$$

Therefore,

$$x_p(t) = -t^2 \int \frac{t\ln t \cdot t^2 \ln t}{t^3}\,dt + t^2 \ln t \int \frac{t\ln t \cdot t^2}{t^3}\,dt = -t^2 \int (\ln t)^2\,dt + t^2 \ln t \int \ln t\,dt\,.$$

Since

$$\int \ln t\,dt = t\ln t - t\,,$$

integrating by parts (with $u = \ln t$ and $dv = \ln t dt$) shows that

$$\begin{aligned}
\int (\ln t)^2\,dt &= \ln t(t\ln t - t) - \int \frac{t\ln t - t}{t}\,dt = t(\ln t)^2 - t\ln t - (t\ln t - t) + t \\
&= t(\ln t)^2 - 2t\ln t + 2t\,,
\end{aligned}$$

we conclude that

$$x_p(t) = -t^2\Big[t(\ln t)^2 - 2t\ln t + 2t\Big] + t^2 \ln t(t\ln t - t) = t^3 \ln t - 2t^3$$

so the general solution to the ODE above is

$$x(t) = C_1 t^2 + C_2 t^2 \ln t + t^3 \ln t - 2t^3\,.$$

By the initial condition $x(1) = -2$ and $x'(1) = -5$, we find that $C_1 = C_2 = 0$; thus the solution to the IVP above is $x(t) = t^3 \ln t - 2t^3$. $\square$