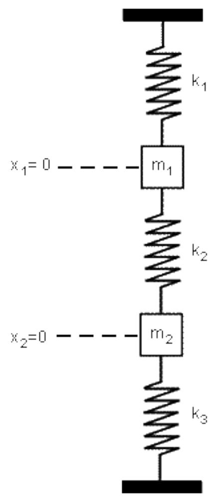
Exercise Problem 1

Due Nov. 14. 2022

Problem 1. Consider the spring-mass system shown in the figure on the right-hand side, where the Hooke constant of the three springs and the mass of two masses are given in the figure. Let $x_1(t)$ and $x_2(t)$ denote the position of masses m_1 and m_2 away form the equilibrium. Assuming the presence of the gravity, suppose that the ODEs that x_1 and x_2 obey are



$$\frac{d^2x_1}{dt^2} = ax_1 + bx_2 + F_1(t)\,,$$
$$\frac{d^2x_2}{dt^2} = cx_1 + dx_2 + F_2(t)\,,$$

Find a, b, c, d and F_1, F_2.

Solution. Let L_1, L_2, L_3 be the length of the unconstraint springs with Hooke's constant k_1, k_2, k_3, respectively, and ℓ_1, ℓ_2, ℓ_3 be the increment of the springs with Hooke's constant k_1, k_2, k_3, respectively. Then

$$k_1\ell_1 = k_2\ell_2 + m_1 g \qquad \text{and} \qquad k_2\ell_2 = k_3\ell_3 + m_2 g\,. \tag{$\star$}$$

Let $x(t), y(t)$ denote the distance, measured from the top wall, of the objects with mass m_1 and m_2, respectively. By Newton's second law of motion, we find that

$$\begin{aligned} m_1\frac{d^2x}{dt^2} &= -k_1(x - L_1) + k_2(y - x - L_2) + m_1 g\,, \\ m_2\frac{d^2y}{dt^2} &= -k_2(y - x - L_2) + k_3(L_1 + \ell_1 + L_2 + \ell_2 + L_3 + \ell_3 - y - L_3) + m_2 g \\ &= -k_2(y - x - L_2) + k_3(L_1 + \ell_1 + L_2 + \ell_2 + \ell_3 - y) + m_2 g\,. \end{aligned}$$

Let x_1 and x_2 denote the position of masses m_1 and m_2 away from the equilibrium. Then

$$x_1 = x - L_1 - \ell_1 \quad \text{and} \quad x_2 = y - L_1 - \ell_1 - L_2 - \ell_2$$

so we have

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= -k_1(x - L_1 - \ell_1 + \ell_1) + k_2\big[(x_2 + L_1 + \ell_1 + L_2 + \ell_2) - (x_1 + L_1 + \ell_1) - L_2\big] + m_1 g \\ &= -k_1x_1 - k_1\ell_1 + k_2(x_2 + \ell_2 - x_1) + m_2 g = -k_1x_1 + k_2(x_2 - x_1) - k_1\ell_1 + k_2\ell_2 + m_1 g\,, \\ m_2\frac{d^2x_2}{dt^2} &= -k_2(y - x - L_2) + k_3(L_1 + \ell_1 + L_2 + \ell_2 + \ell_3 - y) + m_2 g \\ &= -k_2\big[(x_2 + L_1 + \ell_1 + L_2 + \ell_2) - (x_1 + L_1 + \ell_1) - L_2\big] \\ &\quad + k_3\big[L_1 + \ell_1 + L_2 + \ell_2 + \ell_3 - (x_2 + L_1 + \ell_1 + L_2 + \ell_2)\big] + m_2 g \\ &= -k_2(x_2 - x_1 + \ell_2) + k_3(\ell_3 - x_2) + m_2 g = -k_2(x_2 - x_1) - k_3x_2 - k_2\ell_2 + k_3\ell_3 + m_2 g\,. \end{aligned}$$

Using ($\star$), we conclude that x_1 and x_2 satisfy

$$m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2) x_1 + k_2 x_2 \,,$$
$$m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - (k_2 + k_3) x_2 \,.$$

Therefore, $a = -\frac{k_1 + k_2}{m_1}$, $b = \frac{k_2}{m_1}$, $c = \frac{k_2}{m_2}$, $d = -\frac{k_2 + k_3}{m_2}$, and $F_1(t) = F_2(t) = 0$. □

Problem 2. Remove the bottom spring and floor, while x_1 and x_2 still denote the position of masses m_1 and m_2 away from the equilibrium. What are the ODEs that x_1 and x_2 satisfy.

Solution. Removing the bottom spring is the same as setting $k_3 = 0$ (as well as $L_3 = \ell_3 = 0$ in the derivation) in Problem 1. Therefore, x_1 and x_2 satisfy

$$m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2) x_1 + k_2 x_2 \,,$$
$$m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - k_2 x_2 \,.$$

□