Mathematical Modeling MA3067 Take Home Midterm

National Central University, Nov. 6, 2019

Problem 1. (Application of the Pi Theorem)

1. (10%) In tests for fuel economy, cars are driven at constant speed V on a level highway. With no acceleration, the force of propulsion F must be in equilibrium with other forces, such as the air resistance, and so on. Assume that the variables affecting F are the velocity V, the rate C that fuel is burned, in volume per time, and the amount of energy K in a liter of fuel, in mass per length per time-squared. Determine F as a function of V, C and K.

2. (10%) The problem is to determine the power P that must be applied to keep a ship of length ℓ moving at a constant speed V. If it is the case, as seems reasonable, that P depends on the density of water ϱ, the acceleration due to gravity g, and the viscosity of water ν (in length-squared per time), as well as ℓ and V. Show that

$$\frac{P}{\varrho \ell^2 V^3} = f\big(\mathrm{Fr}, \mathrm{Re}\big)\,,$$

   where Fr is the Froude number and Re is the Reynolds number defined by

$$\mathrm{Fr} = \frac{V}{\sqrt{\ell g}}\,, \qquad \mathrm{Re} = \frac{V\ell}{\nu}\,.$$

3. (10%) A chemical **C** flows continuously into a reactor with concentration C_{in} and volumetric flow rate q (volume/time). While in the reactor, which has volume V, the substances are continuously stirred and a chemical reaction

$$\mathbf{C} \;\rightarrow\; \text{products}\,,$$

   with rate constant k (1/time), consumes the chemical. The mixture exits the reactor at the same flow rate q. The concentration of **C** in the reactor at any time is $C = C(t)$, and $C(0) = C_0$. Use dimension analysis to deduce that

$$C = C_{\text{in}} F(tk, a, b)\,,$$

   where $a = C_0/C_{\text{in}}$ and $b = Vk/q$ are dimensionless constants and F is some function.

Problem 2. Consider the ***predator-prey model*** (or the ***Lotka-Volterra model***):

$$\frac{dx}{dt} = \alpha x - \gamma x y\,,$$
$$\frac{dy}{dt} = -\beta y + \delta x y\,,$$

where x and y denote the population of the prey (for example, rabbits) and predator (for example, fox), respectively. A set of initial conditions $x(0) = x_0$ and $y(0) = y_0$ is also given.

1. (15%) Examine the case that $(\alpha, \beta, \gamma, \delta) = (4.5, 0.16, 0.9, 0.08)$ and $(x_0, y_0) = (1, 1000)$ numerically. Is the prey extinct eventually? How about $x_0 = 1$ and $y_0 = 10000$? Does this violate our intuition? Any explanations on this phenomena?

2. (20%) Assume that in addition to the two species, there is a third kind of animals (for example, cows) that appears in this eco-system such that

   (a) The predator cannot consume this new animal (for examples, foxes do not eat cows for the size of cows are much larger than the foxes)

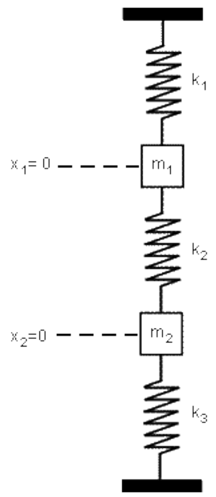
   (b) This new animal compete with the prey in the sense that the whole environment can only supports finite numbers of the prey and the new species (for example, the cows and the rabbits share the same environment and eat the same food). Note that in such a case the environment has finite capacity of the preys and the new animals.

   Suppose the population of this new species is z. Write down (and explain why you have such) a new model for the population of the prey, predator and the new animal. Choose some good parameters (and explain why you choose such parameters) of your model and examine some possible outcomes of your model numerically.

Problem 3. (15%) Consider the spring-mass system shown in the figure on the right-hand side, where the Hooke constant of the three springs and the mass of two masses are given in the figure. Let $x_1(t)$ and $x_2(t)$ denote the position of masses m_1 and m_2 away form the equilibrium. Assuming the presence of the gravity, suppose that the ODE that x_1 and x_2 obeys is

$$\frac{d^2x_1}{dt^2} = ax_1 + bx_2 + F_1(t)\,,$$
$$\frac{d^2x_2}{dt^2} = cx_1 + dx_2 + F_2(t)\,,$$

Find a, b, c, d and F_1, F_2.



Problem 4. Let $\Omega \subseteq \mathbb{R}^3$ be an open set. For a given differentiable vector-valued function

$$\boldsymbol{u} = (u^1, u^2, u^3) : \Omega \to \mathbb{R}^3\,,$$

let curl$\boldsymbol{u}$ be a vector-valued function defined on Ω given by curl$\boldsymbol{u} = \nabla \times \boldsymbol{u}$ or more precisely,

$$(\text{curl}\boldsymbol{u})^i = \sum_{j,k=1}^{3} \varepsilon_{ijk} \frac{\partial u^k}{\partial x_j}\,.$$

Use the tool of the permutation symbols to prove the following two identities.

1. (10%) curlcurl$\boldsymbol{u} = -\Delta \boldsymbol{u} + \nabla$div$\boldsymbol{u}$ if $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ is differentiable, where Δ and div are differentiable operators introduced in the Navier-Stokes equations.

2. (10%) $\boldsymbol{u} \times$ curl$\boldsymbol{u} = \frac{1}{2}\nabla(|\boldsymbol{u}|^2) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ if $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ is differentiable.