NECESSARY DENSITY CONDITIONS FOR SAMPLING AND INTERPOLATION OF CERTAIN ENTIRE FUNCTIONS

BY

H. J. LANDAU

Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey, U.S.A.

Introduction

In a series of seminar lectures given in 1959–60 at the Institute for Advanced Study in Princeton, Professor Arne Beurling posed and discussed the following two problems, in Euclidean spaces:

A. *Balayage.* Let G be a locally compact abelian group, Λ a closed subset of G, and S a given collection of characters. Let $M(G)$ and $M(\Lambda)$ denote the sets of all finite Radon measures having support in G and Λ, respectively. Balayage was said to be possible for S and Λ if corresponding to every $\alpha \in M(G)$ there exists $\beta \in M(\Lambda)$ such that

$$\int \varphi \, d\alpha = \int \varphi \, d\beta, \quad \text{for all} \quad \varphi \in S.$$

Choice of this term was prompted by analogy with its original usage, in which S was a set of potential-theoretic kernels.

The set S, viewed as a subset of the dual group of G, was restricted from the outset to be compact, and to satisfy the regularity conditions (α) and (β) below.

(α) For each $s_0 \in S$ and each neighborhood ω of s_0, there exists a positive Radon measure having support in $\omega \cap S$, with Fourier transform approaching zero at infinity (i.e., outside compact subsets of G).

Let $C(G)$ be the space of bounded continuous functions on G with the uniform norm. Let the weak closure of a set $P \subset C(G)$ consist of those functions of $C(G)$ which are annihilated by every measure in $M(G)$ annihilating P. Let the spectral set Σ_φ of $\varphi \in C(G)$ consist of the characters contained in the weak closure of the set of all translates of φ, and let $C(G, S)$ denote the collection of all $\varphi \in C(G)$ with $\Sigma_\varphi \subset S$.

(β) Spectral synthesis is possible on S (i.e., $C(G, S)$ is contained in the weak closure of the characters in S).

Under these hypotheses, a number of general results were proved regarding balayage, including the following, which establishes a connection with a class of problems in function theory.

THEOREM. *Balayage is possible for S and Λ if and only if there exists a constant $K(S, \Lambda)$ such that for all $\varphi \in C(G, S)$*

$$\sup_{x \in G} |\varphi(x)| \leqslant K(S, \Lambda) \sup_{x \in \Lambda} |\varphi(x)|.$$

Attention was then focused on the case that G is the real line and S a single interval, and an explicit solution to the problem of balayage was given. A subset Λ_0 of the reals was termed uniformly discrete if the distance between any two distinct points of Λ_0 exceeds some positive quantity. For such Λ_0, let $n^+(r)$, $n^-(r)$ denote respectively the largest and smallest number of points of Λ_0 to be found in an interval of length r; the limits

$$D^+(\Lambda_0) = \lim_{r \to \infty} n^+(r)/r \quad \text{and} \quad D^-(\Lambda_0) = \lim_{r \to \infty} n^-(r)/r,$$

which always exist, were called the upper and lower uniform densities of Λ_0.

THEOREM A. *When S is a single interval of the real line, balayage is possible for S and Λ if and only if Λ contains a uniformly discrete subset Λ_0 with $D^-(\Lambda_0) > \text{measure}(S)/2\pi$.*

B. *Interpolation.* Interpolation was said to be possible for S and Λ if for every $f \in C(G)$ there exists $\varphi \in C(G, S)$ such that $\varphi(x) = f(x)$, for $x \in \Lambda$. This problem was introduced as a dual of the problem of balayage. Once again, after some preliminary general results, the discussion was specialized to the case that S is an interval of the real line, and the following solution was given.

THEOREM B. *When S is a single interval of the real line, interpolation is possible for S and Λ if and only if Λ is uniformly discrete and $D^+(\Lambda) < \text{measure}(S)/2\pi$.*

Theorems A and B were proved by complex-variable methods.

These striking characterizations cannot be expected to persist when the interval S is replaced by a more complicated set, since arithmetic relations among the points of Λ then play an important role. Nevertheless, Beurling conjectured that the density condition of Theorem A remains necessary for balayage. One of the main objects of this paper is to establish that conjecture in R^N.

We proceed by considering, in R^N, L^2 versions of the above problems. Thus with S

a measurable set in R^N, we let $\mathcal{B}(S)$ denote the subspace of $L^2(R^N)$ consisting of those functions whose Fourier transform is supported on S, and let $\|f\|$ denote the L^2 norm of f. We say that a subset Λ of R^N is uniformly discrete if the distance between any two distinct points of Λ exceeds some positive quantity, which we term the separation of Λ. We call a uniformly discrete Λ a *set of sampling for* $\mathcal{B}(S)$ if there exists a constant K such that $\|f\|^2 \leqslant K \sum_{\lambda\in\Lambda} |f(\lambda)|^2$ for every $f \in \mathcal{B}(S)$, and a *set of interpolation for* $\mathcal{B}(S)$ if, corresponding to each square-summable collection of complex numbers $\{a_\lambda\}_{\lambda\in\Lambda}$, there exists $f \in \mathcal{B}(S)$ with $f(\lambda) = a_\lambda$, $\lambda \in \Lambda$; similar sets were considered in [1] and [6] under the names of "frames" and "Riesz-Fischer sequences", respectively. For bounded S, the members of $\mathcal{B}(S)$ are restrictions to R^N of certain entire functions of N complex variables. When S is a single interval in R^1, $\mathcal{B}(S)$ has been extensively studied, mainly by function-theoretic methods. These methods largely fail already when S is replaced by the union of several intervals, and possess no counterparts at all in more than one dimension. Here we will connect sets of sampling and interpolation with an eigenvalue problem which is tractable independently of the dimension, and by exploiting the relationship will obtain information about each. In particular, in R^1 this approach will show:

THEOREM 1. *If S is the union of a finite number of intervals of total measure $m(S)$, and Λ is a set of sampling for $\mathcal{B}(S)$,*

$$n^-(r) \geqslant (2\pi)^{-1} m(S)\, r - A \log^+ r - B,$$

with constants A and B independent of r.

THEOREM 2. *If S is as in Theorem 1, and Λ is a set of interpolation for $\mathcal{B}(S)$,*

$$n^+(r) \leqslant (2\pi)^{-1} m(S)\, r + A \log^+ r + B,$$

with constants A and B independent of r.

With S an arbitrary bounded and measurable set in R^N, our conclusions will be somewhat less precise, in that the above bounds are replaced by asymptotic versions; nevertheless they will be sufficient to establish sharp density-measure theorems for sets of sampling or interpolation, and to prove Beurling's conjecture regarding balayage. They also show that sets of sampling for $\mathcal{B}(S)$ differ greatly from sets of uniqueness, whose density need bear no relation to the measure of S [2].

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Preliminaries

Let R^N be N-dimensional Euclidean space. We denote by x a point $(x_1, ..., x_N)$ of R^N and by dx the Lebesgue product measure $dx_1, ..., dx_N$. If S is a subset of R^N, τ a point of R^N, and r a positive scalar, we denote by $S+\tau$ the translate of S by τ, i.e., the set of points of R^N of the form $x+\tau$ with $x\in S$, and by rS the scaling of S by r, i.e., the set of points of R^N of the form rx with $x\in S$. Let m represent Lebesgue measure in R^N; then for measurable sets S, $m(rS)=r^N m(S)$. If $x=(x_1, ..., x_N)$ and $y=(y_1, ..., y_N)$ are two points of R^N, we let

$$|x|' = (x_1^2 + ... + x_N^2)^{\frac{1}{2}} \quad \text{and} \quad xy = x_1 y_1 + ... + x_N y_N.$$

Henceforth we will use the term "set" to mean "measurable set".

The square-integrable functions on R^N form a Hilbert space with scalar product

$$(f,g) = \int_{R^N} f(x)\overline{g(x)}\,dx;$$

we shall write $f\perp g$ if $(f,g)=0$. In this space the Fourier transform T, defined by

$$Tf = (2\pi)^{-N/2}\int_{R^N} f(y)\,e^{-ixy}\,dy,$$

is a unitary operator with inverse

$$T^{-1}f = (2\pi)^{-N/2}\int_{R^N} f(y)\,e^{ixy}\,dy.$$

If P is a set in R^N, we will denote by χ_P both the characteristic function of P, i.e., the function whose values are 1 on P and 0 elsewhere, and the operator in $L^2(R^N)$ defined by

$$\chi_P f = \chi_P(x)\,f(x).$$

An eigenvalue problem

With Q and S two sets in R^N, let $\mathcal{D}(Q)$ be the subspace of $L^2(R^N)$ consisting of those functions supported on Q, and $\mathcal{B}(S)$ be as previously defined. Let D_Q and B_S denote the orthogonal projections of $L^2(R^N)$ onto $\mathcal{D}(S)$ and $\mathcal{B}(Q)$, respectively; they are given explicitly by

$$B_S = T^{-1}\chi_S T, \tag{1}$$

$$D_Q = \chi_Q. \tag{2}$$

In the following lemma we collect some elementary properties of the operator $B_S D_Q B_S$, which takes $\mathcal{B}(S)$ into itself.

LEMMA 1. *If the sets S and Q have finite measure, the bounded self-adjoint positive operator $B_S D_Q B_S$ is completely continuous. Denoting its eigenvalues, arranged in nonincreasing order, by $\lambda_k(S, Q)$, $k = 0, 1, \ldots$, we find for all k*

(i) $\lambda_k(S, Q) = \lambda_k(S + \sigma, Q + \tau) = \lambda_k(\alpha S, \alpha^{-1} Q)$, for any $\sigma, \tau \in R^N$ and $\alpha > 0$,

(ii) $\lambda_k(S, Q) = \lambda_k(Q, S)$,

(iii) $\sum_k \lambda_k(S, Q) = (2\pi)^{-N} m(S)\, m(Q)$,

(iv) $\sum_k \lambda_k^2(S, Q) \geqslant \sum_k \lambda_k^2(S, Q_1) + \sum_k \lambda_k^2(S, Q_2)$, *if* $Q = Q_1 \cup Q_2$ *with* Q_1 *and* Q_2 *disjoint,*

(v) $\sum_k \lambda_k^2(S, Q) \geqslant \{(2\pi)^{-1} sq - \pi^{-2} \log^+ sq - 1\}^N$, *if S and Q are cubes with edges parallel to the coordinate axes, of volumes s^N and q^N respectively,*

(vi) $\lambda_k(S, Q_1) \leqslant \lambda_k(S, Q_2)$, *if* $Q_1 \subset Q_2$,

(vii) $\lambda_k(S, Q) \leqslant \sup_{f \in B(S),\, f \perp C_k} \|D_Q f\|^2 / \|f\|^2$,

(viii) $\lambda_{k-1}(S, Q) \geqslant \inf_{f \in B(S),\, f \in C_k} \|D_Q f\|^2 / \|f\|^2$, *if C_k is any k-dimensional subspace of $L^2(R^N)$.*

Proof. Since projections are bounded by 1, self-adjoint, and idempotent,

$$(B_S D_Q B_S f, f) = \|D_Q B_S f\|^2 \leqslant \|f\|^2, \tag{3}$$

so that $B_S D_Q B_S$ is bounded by 1, self-adjoint, and positive. Using (1) and (2), $D_Q B_S$ may (as below) be written explicitly as an integral operator whose kernel is square-integrable (by Parseval's theorem), hence [5, p. 158] $D_Q B_S$, so also $B_S D_Q B_S$ and $D_Q B_S D_Q$, are completely continuous. Let us write $A \sim B$ if the completely continuous operators A and B have the same nonzero eigenvalues, including multiplicities. If $B_S D_Q B_S \varphi = \lambda \varphi$, and $\lambda \neq 0$, then $\varphi = B_S \varphi$ since B_S is a projection, and $\|D_Q B_S \varphi\| \neq 0$. An application of the projection D_Q to the equation now yields $D_Q B_S D_Q (D_Q B_S \varphi) = \lambda D_Q B_S \varphi$, so that λ is likewise an eigenvalue of $D_Q B_S D_Q$. The same argument in the other direction shows

$$B_S D_Q B_S \sim D_Q B_S D_Q. \tag{4}$$

Because the spectrum is real, $D_Q B_S D_Q \sim C D_Q B_S D_Q C$ where C denotes complex conjugation, and since C commutes with χ and $CTC = T^{-1}$, by (1) and (2) $C D_Q B_S D_Q C = \chi_Q T \chi_S T^{-1} \chi_Q$. Finally, T is unitary, so that $\chi_Q T \chi_S T^{-1} \chi_Q \sim T^{-1} \chi_Q T \chi_S T^{-1} \chi_Q T = B_Q D_S B_Q$. Combining these, we conclude $B_S D_Q B_S \sim B_Q D_S B_Q$, proving (ii). By (1) and (2) the operator $D_Q B_S D_Q$ may be written explicitly as

$$D_Q B_S D_Q f = (2\pi)^{-N/2} \int_{R^N} \chi_Q(x)\, \chi_Q(y)\, k(y - x)\, f(x)\, dx,$$

where Tk coincides with $\chi_S(x)$. From (4) and a change of variable we obtain (i), and on further applying known results [5, p. 243–5] to this representation we find

$$\textstyle\sum_k \lambda_k(S,Q) = (2\pi)^{-N/2}\int_{R^N} \chi_Q(x)\, k(0)\, dx = (2\pi)^{-N} m(S)\, m(Q),$$

establishing (iii), and

$$\textstyle\sum_k \lambda_k^2(S,Q) = (2\pi)^{-N}\iint_{Q\times Q} |k(y-x)|^2 dx\, dy. \tag{5}$$

Now if $Q = Q_1 \cup Q_2$ with Q_1 and Q_2 disjoint, the set $Q \times Q$ over which the integral in (5) is extended includes $Q_1 \times Q_1 \cup Q_2 \times Q_2$, and since the integrand is nonnegative, (iv) follows. To estimate (5) for the case that S and Q are cubes with edges parallel to the coordinate axes, of volumes s^N and q^N respectively, we may assume by (i) that the centers are at the origin, whereupon

$$k(y) = \prod_{i=1}^{N} (2/\pi)^{\frac{1}{2}} (y_i)^{-1} \sin \tfrac{1}{2} s y_i,$$

with y_i the ith coordinate of y, and

$$\textstyle\sum_k \lambda_k^2(S,Q) = \left\{ \pi^{-2} \int_{|u|<q/2} \int_{|v|<q/2} \sin^2 \tfrac{1}{2}s(u-v)/(u-v)^2 du\, dv \right\}^N,$$

where u and v are one-dimensional variables. Applying a change of variable, the identity

$$\int_0^\infty \sin^2 t/t^2 dt = \pi/2,$$

and some manipulation, we obtain (v). Next we invoke the Weyl-Courant Lemma [5, p. 238] to prove (vi), since $B_S D_{Q_2} B_S$ and $B_S D_{Q_1} B_S$ differ by the positive operator $B_S D_{(Q_2 - Q_1)} B_S$, and to establish the bounds

$$\lambda_k(S,Q) \leqslant \sup_{f \perp C_k} (B_S D_Q B_S f, f)/\|f\|^2, \tag{6}$$

$$\lambda_{k-1}(S,Q) \geqslant \inf_{f \in C_k} (B_S D_Q B_S f, f)/\|f\|^2. \tag{7}$$

If $C_k \subset \mathcal{B}(S)$, the requirement $f \in C_k$ of (7) implies $f = B_S f$, thus as in (3) we find (viii). The subspace $B_S C_k$ of $\mathcal{B}(S)$ has dimension $d \leqslant k$, $f \perp B_S C_k$ if and only if $B_S f \perp C_k$, and $\|g\|^2 \geqslant \|B_S g\|^2$, so by (6) the right-hand side of (vii) is an upper bound for $\lambda_d(S,Q)$, hence also for $\lambda_k(S,Q)$. This completes the proof of Lemma 1.

In view of (iii) it is natural to inquire how the values of $\lambda_k(S,Q)$ are distributed. A qualitative description, at least when S and Q are sufficiently regular, is that $\lambda_k(S,Q)$ is very close to 1, then very close to zero, the transition occurring in a relatively narrow range of k, centered at $2\pi^{-N} m(S)\, m(Q)$. This fact, which we shall demonstrate more precisely, will play an important role in our argument.

Sets of sampling and interpolation

With a uniformly discrete set Λ we may associate a counting function n, defined on compact subsets I or R^N as the number of points of Λ contained in I. We proceed to establish a connection between the counting function of a set of sampling or interpolation and the behavior of certain of the eigenvalues introduced in the last section. We may perhaps account intuitively for this connection by viewing both the eigenvalues and the counting functions as describing the number of functions of $\mathcal{B}(S)$ which are in a sense independent and well-concentrated on a given set.

LEMMA 2. *Let S be a bounded set and Λ a set of sampling for $\mathcal{B}(S)$, with separation d and counting function n. Let I be any compact set, and $I+$ be the set of points whose distance to I is less than $d/2$. Then $\lambda_{n(I+)}(S, I) \leqslant \gamma < 1$, where γ depends on S and Λ but not on I.*

Proof. We choose $h(y) \in L^2(R^N)$ so that $h(y)$ vanishes for $|y|' > d/2$ and its Fourier transform Th satisfies $|Th| \geqslant 1$ for $x \in S$. Such a choice is possible since Fourier transforms of functions supported on $|y|' < d/2$ are uniformly dense in continuous functions on any bounded set. Given $f \in \mathcal{B}(S)$ we form

$$g(x) = (2\pi)^{-N/2} \int_{R^N} f(y)\, h(x-y)\, dy = (2\pi)^{-N/2} \int_{|x-y|' < d/2} f(y)\, h(x-y)\, dy, \tag{8}$$

whence $Tg = (Tf)(Th)$, so that $g \in \mathcal{B}(S)$. Since Tf vanishes outside S, we find by Parseval's theorem and the definition of h

$$\|g\|^2 = \|Tg\|^2 = \|(Tf)(Th)\|^2 \geqslant \|Tf\|^2 = \|f\|^2, \tag{9}$$

and by Schwarz's inequality applied to (8)

$$|g(x)|^2 \leqslant (2\pi)^{-N} \|h\|^2 \int_{|x-y|' < d/2} |f(y)|^2 dy. \tag{10}$$

Finally, by definition of Λ, since $g \in \mathcal{B}(S)$,

$$\|g\|^2 \leqslant K \textstyle\sum_{\lambda \in \Lambda} |g(\lambda)|^2. \tag{11}$$

Now let $\mathcal{C}$ be the subspace of $L^2(R^N)$ spanned by the functions $\overline{h(\lambda - x)}$ for $\lambda \in \Lambda \cap I+$; because these functions are all orthogonal, the dimension of $\mathcal{C}$ is $n(I+)$. If $f \in \mathcal{B}(S)$ and $f \perp \mathcal{C}$, we see from (8) that $g(\lambda) = 0$ for $\lambda \in \Lambda \cap I+$. Thereupon, combining (9), (11), and (10),

$$\begin{aligned}
\|f\|^2 &\leqslant \|g\|^2 \leqslant K \textstyle\sum_{\lambda \in \Lambda,\, \lambda \notin I+} |g(\lambda)|^2 \\
&\leqslant K(2\pi)^{-N} \|h\|^2 \textstyle\sum_{\lambda \in \Lambda,\, \lambda \notin I+} \displaystyle\int_{|y-\lambda|' < d/2} |f(y)|^2 dy \\
&\leqslant K(2\pi)^{-N} \|h\|^2 \int_{y \notin I} |f(y)|^2\, dy = K(2\pi)^{-N} \|h\|^2 \left[\|f\|^2 - \int_I |f(y)|^2 dy \right],
\end{aligned}$$

hence $\|D_I f\|^2/\|f\|^2 \leqslant 1-(2\pi)^N K^{-1}\|h\|^{-2}=\gamma<1$. Applying Lemma 1 (vii), we conclude that $\lambda_{n(I+)}(S, I)\leqslant\gamma<1$. The constant γ depends on S and Λ since K and $\|h\|^2$ do, but does not depend on I. Lemma 2 is established.

LEMMA 3. *Let S be a bounded set and Λ a set of interpolation for $\mathcal{B}(S)$, with separation d and counting function n. Let I be any compact set, and $I-$ be the set of points whose distance to the complement of I exceeds $d/2$. Then $\lambda_{n(I-)-1}(S, I)\geqslant\delta>0$, where δ depends on S and Λ but not on I.*

Proof. Let h be the function introduced in Lemma 2. If $g\in\mathcal{B}(S)$, choosing f to be the function of $\mathcal{B}(S)$ whose Fourier transform is $(Tg)/(Th)$ yields (8), whereupon from (10) and (9), $\sum_{\lambda\in\Lambda}|g(\lambda)|^2\leqslant(2\pi)^{-N}\|h\|^2\|f\|^2\leqslant(2\pi)^{-N}\|h\|^2\|g\|^2$. Consequently, whenever Λ is uniformly discrete, the mapping M given by $f\to\{f(\lambda)\}_{\lambda\in\Lambda}$ is a bounded transformation of $\mathcal{B}(S)$ into l^2. Now let $\mathcal{E}^0(S)$ be the subspace of $\mathcal{B}(S)$ consisting of all $f\in\mathcal{B}(S)$ which vanish at the points of Λ. Schwarz's inequality and Parseval's theorem applied to the representation of $f\in\mathcal{B}(S)$ as the inverse transform of its Fourier transform show that $|f(y)|^2\leqslant 2\pi^{-N}m(S)\|f\|^2$, hence in $\mathcal{B}(S)$ convergence in norm implies uniform convergence, so that $\mathcal{E}^0(S)$ is a closed subspace. We denote by $\mathcal{E}(S)$ the orthogonal complement in $\mathcal{B}(S)$ of $\mathcal{E}^0(S)$, and observe that Λ is a set of interpolation also for $\mathcal{E}(S)$; thus M effects a one-to-one linear mapping of $\mathcal{E}(S)$ onto l^2. By a theorem of Banach [4, p. 18], this mapping has a bounded inverse, hence there exists a constant K satisfying

$$\|g\|^2\leqslant K\sum_{\lambda\in\Lambda}|g(\lambda)|^2 \tag{12}$$

for every $g\in\mathcal{E}(S)$.

For each $\lambda\in\Lambda$, let $\varphi_\lambda(x)\in\mathcal{E}(S)$ be the function whose value is 1 at λ and 0 at every other point of Λ; these functions are linearly independent. As above, we construct $\psi_\lambda\in\mathcal{B}(S)$ so that

$$\varphi_\lambda(x)=(2\pi)^{-N/2}\int_{R^N}\psi_\lambda(y)\,h(x-y)\,dy; \tag{13}$$

the ψ_λ are likewise linearly independent. We let C be the subspace of $\mathcal{B}(S)$ spanned by the functions ψ_λ for $\lambda\in\Lambda\cap I-$; the dimension of C is then $n(I-)$. Now given $f\in\mathsf{C}$, we form the function g of (8), whence by definition of $I-$ and (10),

$$(2\pi)^{-N}\|h\|^2\int_I|f(y)|^2dy\geqslant(2\pi)^{-N}\|h\|^2\sum_{\lambda\in\Lambda\cap I-}\int_{|y-\lambda|'<d/2}|f(y)|^2dy\geqslant\sum_{\lambda\in\Lambda\cap I-}|g(\lambda)|^2. \tag{14}$$

But by (13), g is a linear combination of the φ_λ, hence is in $\mathcal{E}(S)$, so combining (14), (12), and (9) we find, for $f\in\mathsf{C}$, $\|D_I f\|^2/\|f\|^2\geqslant(2\pi)^N K^{-1}\|h\|^{-2}=\delta>0$. Applying Lemma 1 (viii), we conclude that $\lambda_{n(I-)-1}(S, I) \geq \delta > 0$. The constant δ depends on S and Λ since K and $\|h\|^2$ do, but does not depend on I. Lemma 3 is established.

Taken together, Lemmas 2 and 3 may be exploited in two ways. When applied to uniformly distributed sets Λ which are known to be sets of sampling or interpolation for $\mathcal{B}(S)$, they give bounds on certain eigenvalues of $B_S D_I B_S$; some of these results were developed in [3]. With this information, they may in turn be used to compare the behavior of arbitrary sets of sampling or interpolation with that of known ones.

A special case

THEOREM 1. *Let S be the union of a finite number of intervals in R^1, and Λ be a set of sampling for $\mathcal{B}(S)$. With I the unit interval centered at the origin, set $n^-(r) = \min_{\tau \in R^1} n(rI + \tau)$. Then*

$$n^-(r) \geq (2\pi)^{-1} m(S)\, r - A \log^+ r - B,$$

where the constants A and B depend on S and Λ but not on r.

Proof. Let $\mathcal{S}$ be an interval of length r such that $n^-(r) = n(\mathcal{S})$. Since $\mathcal{S}$ is a single interval, $n(\mathcal{S}+) \leq n(\mathcal{S}) + 2$, hence by Lemma 1 (i) and Lemma 2

$$\lambda_{n(\mathcal{S})+2}(S, rI) \leq \lambda_{n(\mathcal{S}+)}(S, \mathcal{S}) \leq \gamma < 1, \tag{15}$$

with γ independent of r.

Suppose S to consist of p disjoint intervals $S_1, \ldots, S_p$; denoting by l_i the length of S_i, we have $\sum l_i = m(S)$. By Lemma 1 (ii) and (i), $\lambda_k(S, rI) = \lambda_k(rI, S) = \lambda_k(2\pi I, 2\pi^{-1} rS)$. Now the value of a function $f \in \mathcal{B}(2\pi I)$ at an integer $x = -k$ coincides with the kth Fourier coefficient of Tf; hence by the Parseval and Riesz-Fischer theorems, the set Λ^* consisting of the integers is a set both of sampling and interpolation for $\mathcal{B}(2\pi I)$. The set $2\pi^{-1} rS$ consists of p disjoint intervals; it is not hard to see that the number of integers contained in $\{(2\pi^{-1} rS)-\}$ exceeds $(2\pi)^{-1} r m(S) - 2p$ [3, Lemma 1]. We may now apply Lemma 3 with Λ^* to conclude

$$\lambda_{[2\pi^{-1} r m(S)] - 2p - 1}(S, rI) = \lambda_{[2\pi^{-1} r m(S)] - 2p - 1}(2\pi I, 2\pi^{-1} rS) \geq \delta > 0, \tag{16}$$

where δ is independent of r, and $[\eta]$ denotes the integer part of η.

To be able to compare the indices of the eigenvalues appearing in (15) and (16), we require an estimate of the number of $\lambda_k(S, rI)$ which are not near 1 or near 0. Accordingly, we consider

$$J(S, rI) = \sum_k \lambda_k(S, rI)\{1 - \lambda_k(S, rI)\}.$$

By Lemma 1 (iii) and (iv),

$$J(S, rI) \leq (2\pi)^{-1} m(S)\, r - \sum_{i=1}^{p} \sum_k \lambda_k^2(S_i, rI),$$

and since S_i and rI are intervals of lengths l_i and r we may apply Lemma 1 (v), with $N=1$, to obtain

$$J(S, rI) \leqslant (2\pi)^{-1}m(S)r - \sum_i (2\pi)^{-1} r l_i + \pi^{-2}\sum_i \log^+ r l_i + p \leqslant A' \log^+ r + B',$$

where A' and B' are constants depending only on S. Now if $n(\mathcal{S})+2 \leqslant [2\pi^{-1} r m(S)] - 2p - 1$, then for every k in the range $n(\mathcal{S})+2 \leqslant k \leqslant [2\pi^{-1} r m(S)] - 2p - 1$, we see by (15) and (16) that $0 < \delta \leqslant \lambda_k(S, rI) \leqslant \gamma < 1$, so that the contribution to J from each of these eigenvalues is at least $\alpha = \min\{\delta(1-\delta), \gamma(1-\gamma)\} > 0$. Hence

$$\{[2\pi^{-1} r m(S)] - 2p - n(\mathcal{S}) - 2\}\alpha \leqslant J(S, rI) \leqslant A' \log^+ r + B',$$

or

$$n(\mathcal{S}) \geqslant (2\pi)^{-1} m(S) r - \frac{A'}{\alpha} \log^+ r - \frac{B'}{\alpha} - 2p - 3.$$

If $n(\mathcal{S})+2 > [2\pi^{-1} r m(S)] - 2p - 1$ the above inequality holds a fortiori. Setting $A = A'/\alpha$ and $B = B'/\alpha + 2p + 3$, and using the definition of $\mathcal{S}$, we obtain

$$n^-(r) \geqslant (2\pi)^{-1} m(S) r - A \log^+ r - B,$$

with constants A and B depending on S and Λ but not on r. Theorem 1 is established.

THEOREM 2. *Let S be the union of a finite number of intervals in R^1, and Λ be a set of interpolation for $\mathcal{B}(S)$. With I the unit interval centered at the origin, set $n^+(r) = \max_{\tau \in R^1} n(rI + \tau)$. Then*

$$n^+(r) \leqslant (2\pi)^{-1} m(S) r + A \log^+ r + B,$$

where the constants A and B depend on S and Λ but not on r.

Proof. We let $\mathcal{S}$ be an interval of length r such that $n^+(r) = n(\mathcal{S})$, and follow the proof of Theorem 1, interchanging the roles of Lemmas 2 and 3. Thus since $\mathcal{S}$ is a single interval, $n(\mathcal{S}-) \geqslant n(\mathcal{S}) - 2$, so that $\lambda_{n(\mathcal{S})-3}(S, rI) \geqslant \delta > 0$. Then since the integers are a set of sampling for $\mathcal{B}(2\pi I)$ and since their number in $\{(2\pi^{-1} r S)+\}$ can easily be shown not to exceed $[(2\pi)^{-1} r m(S)] + 2p$ we see that $\lambda_{[2\pi^{-1} r m(S)]+2p}(S, rI) \leqslant \gamma < 1$. Then if $[2\pi^{-1} r m(S)] + 2p \leqslant n(\mathcal{S}) - 3$, we again consider the contribution to $J(S, rI)$ of the intermediate eigenvalues, obtaining $n(\mathcal{S}) \leqslant (2\pi)^{-1} r m(S) + A'/\alpha \log^+ r + B'/\alpha + 2p + 2$, an inequality which holds a fortiori if $[2\pi^{-1} r m(S)] + 2p > n(\mathcal{S}) - 3$. Letting $A = A'/\alpha$ and $B = B'/\alpha + 2p + 1$ we find

$$n^+(r) \leqslant (2\pi)^{-1} m(S) r + A \log^+ r + B,$$

with constants A and B depending on S and Λ but not on r. Theorem 2 is established.

Upper and lower density

In R^1 we measured density of a uniformly discrete set Λ in terms of the functions $n^{\pm}(r)=n^{\pm}(rI)$, with I the unit interval. In more than one dimension, the choice of I presents an additional element of freedom: corresponding to each I of measure 1 we could consider $n^+(rI)$ and $n^-(rI)$, defined as the largest and smallest number of points of Λ to be found in a translate of rI, and, following Beurling, speak of upper and lower uniform densities of Λ,

$$D^+(I,\Lambda)=\limsup_{r\to\infty} n^+(rI)/r^N \quad \text{and} \quad D^-(I,\Lambda)=\liminf_{r\to\infty} n^-(rI)/r^N.$$

Our first aim is to show that, under mild regularity conditions, these densities do not depend on I; this will entitle us to refer unambiguously to the upper and lower uniform densities of Λ, and to evaluate them by means of cubes.

LEMMA 4. *In R^N, let Λ be a uniformly discrete set, and U be the closed unit cube with sides parallel to the coordinate axes, centered at the origin. If I is a compact set of measure* 1 *whose boundary has measure* 0, *then*

$$D^+(I,\Lambda)=D^+(U,\Lambda) \quad \textit{and} \quad D^-(I,\Lambda)=D^-(U,\Lambda).$$

Proof. Let I' and U' denote the interiors of I and U, respectively; they are open sets of measure 1. Since $m(I)=1$, I may be covered by a countable union of scaled translates of U', having measure arbitrarily close to 1. Because I is compact, there exists a finite subcovering, each cube of which may, at the cost of another arbitrarily small increase of measure, be included in a cube similarly oriented but with edge of rational length and center with rational coordinates. Since any finite number of rationals may be written with a common denominator, the closure of this covering may be expressed as the union of a finite number of scaled translates of U, whose interiors are disjoint. The same construction applied to the complement, inside some large cube, of I', followed by complementation, shows that there exist in I' finite collections of scaled translates of U having disjoint interiors and total measure arbitrarily close to $m(I')=1$. Contracting each of these cubes by a sufficiently small amount yields a disjoint union of scaled translates of U, contained in I, with measure arbitrarily close to 1.

We may similarly approximate U from inside and outside by finite unions of scaled translates of I. For since I is compact, there exists $\varrho>0$ such that a translate of ϱI may be contained in U'; the remainder R_1 of U' is an open set of measure $1-\varrho^N$, having compact closure $\bar{R}_1$ and boundary of measure 0. With $\alpha<1$, by the last construction we may find in R_1 a finite union of disjoint scaled translates of U, of total measure $\alpha(1-\varrho^N)$; taking from within each of these cubes the correspondingly scaled translate of ϱI, we obtain a finite disjoint collection of the latter of total measure $\varrho^N\alpha(1-\varrho^N)$, and a remaining open set $R_2 \subset U'$ of measure $(1-\alpha\varrho^N)(1-\varrho^N)$, with $\bar{R}_2$ compact and $m(R_2)=m(\bar{R}_2)$. Iterating this construction, we find that R_k, the subset of U' left uncovered at the kth step, is an open set with compact closure $\bar{R}_k$, and $m(R_k)=m(\bar{R}_k)=(1-\varrho^N)(1-\alpha\varrho^N)^{k-1}$, which approaches 0 as $k\to\infty$. Hence we may find in U a finite union of disjoint scaled translates of I, with measure arbitrarily close to 1. To cover U, we cover $\bar{R}_k$, as in the first part of the argument, by a finite union of scaled translates of U, having disjoint interiors and total measure m' arbitrarily close to $m(\bar{R}_k)$. Let $\sigma>0$ be such that I' contains a translate of σU; since $m(I')=1$, the above union of cubes can be included in a finite union of scaled translates of I, the sum of whose measures is m'/σ^N. Thus U is included in a finite union of scaled translates of I, the sum of whose measures is $1-m(R_k)+m'/\sigma^N$, which can be made arbitrarily close to 1 by taking $m(R_k)$ and m' sufficiently small.

Now given $\varepsilon>0$, let $\{\varrho_i U+\tau_i\}$ form a finite collection of disjoint subsets of I having total measure $\sum_i \varrho_i^N > 1-\varepsilon$. Then $n^-(rI)\geqslant \sum_i n^-(r\varrho_i U)$, so that

$$n^-(rI)\,r^{-N} \geqslant \sum_i n^-(r\varrho_i U)(r\varrho_i)^{-N}\varrho_i^N,$$

whence $D^-(I,\Lambda)\geqslant D^-(U,\Lambda)(1-\varepsilon)$. Since ε is arbitrary, $D^-(I,\Lambda)\geqslant D^-(U,\Lambda)$, and the same argument with the roles of U and I interchanged shows the reverse inequality; thus $D^-(I,\Lambda)=D^-(U,\Lambda)$. The analogous argument, using coverings of I and U by scaled translates of U and I, respectively, shows that $D^+(I,\Lambda)=D^+(U,\Lambda)$. Lemma 4 is established.

We remark that for integral k and any r, $n^+(krU)\leqslant k^N n^+(rU)$ and $n^-(krU)\geqslant k^N n^-(rU)$; it follows that the quantities $n^\pm(rU)r^{-N}$ appearing in the definitions of $D^\pm(U,\Lambda)$ both have limits as $r\to\infty$. We henceforth abbreviate these densities by $D^\pm(\Lambda)$.

The general case

When $S\in R^N$ is the union of a finite number of disjoint scaled translates of U, the proofs of Theorems 1 and 2 can be repeated without change to yield bounds of the form

$$n^-(rU)\geqslant(2\pi)^{-N}m(S)\,r^N - Ar^{N-1}\log^+ r - B$$

and

$$n^+(rU)\leqslant(2\pi)^{-N}m(S)\,r^N + Ar^{N-1}\log^+ r + B$$

for sets of sampling and interpolation, respectively. For more general sets S we apply a process of approximation to obtain asymptotic versions of these bounds, which we express in terms of the upper and lower density.

THEOREM 3. *Let S be a set in R^N and Λ be a set of sampling for $\mathcal{B}(S)$. Then* $D^-(\Lambda) \geqslant (2\pi)^{-N} m(S)$.

Proof. It is sufficient to prove the theorem for compact sets S. For every S has compact subsets S_n with $m(S_n)$ arbitrarily close to $m(S)$ (arbitrarily large when $m(S)=\infty$), and Λ remains a set of sampling for $\mathcal{B}(S_n)$. Applying the result to S_n we find $D^-(\Lambda) \geqslant (2\pi)^{-N} m(S_n)$, whence, letting $n \to \infty$, $D^-(\Lambda) \geqslant (2\pi)^{-N} m(S)$.

Let $\mathcal{U}$ be a translate of rU such that

$$n(\mathcal{U}) = n^-(rU) = \min_{\tau \in R^N} n(rU+\tau).$$

By Lemma 1 (i) and Lemma 2,

$$\lambda_{n(\mathcal{U}+)}(S, rU) = \lambda_{n(\mathcal{U}+)}(S, \mathcal{U}) \leqslant \gamma < 1, \tag{17}$$

with γ independent of r. By Lemma 1 (ii) and (i), for all k,

$$\lambda_k(S, rU) = \lambda_k(rU, S) = \lambda_k(2\pi U, 2\pi^{-1} rS). \tag{18}$$

Let Λ^* be the lattice of points in R^N all of whose coordinates are integers, and n^* be the counting function of Λ^*. Exactly as in R^1, Λ^* is a set of sampling and interpolation for $\mathcal{B}(2\pi U)$, but we cannot obtain a serviceable lower bound for $\lambda_k(S, rU)$ directly from Lemma 3, since $(2\pi^{-1}rS)-$ may be empty. Instead, given $\varepsilon > 0$, we cover S, as in the proof of Lemma 4, by a finite collection S_ε of scaled translates of U, having disjoint interiors and $m(S_\varepsilon) < m(S)+\varepsilon$. By (18) and Lemma 3 applied to Λ^* and $\mathcal{B}(2\pi U)$,

$$\lambda_{n^*\{(2\pi^{-1}rS_\varepsilon)-\}-1}(S_\varepsilon, rU) \geqslant \delta > 0 \tag{19}$$

with δ independent of r. Let $J(S_\varepsilon, rU) = \sum_k \lambda_k(S_\varepsilon, rU)\{1-\lambda_k(S_\varepsilon, rU)\}$, and let M be the number of eigenvalues $\lambda_k(S_\varepsilon, rU)$ which exceed $(1+\gamma)/2$. If the index of the eigenvalue of (19) exceeds M, then for every k between the two, the contribution to $J(S_\varepsilon, rU)$ of the corresponding $\lambda_k(S_\varepsilon, rU)$ is at least $\alpha = \min\{\delta(1-\delta), (1-\gamma)^2/4\} > 0$; thus

$$n^*\{(2\pi^{-1}rS_\varepsilon)-\} - M \leqslant \alpha^{-1} J(S_\varepsilon, rU), \tag{20}$$

an inequality which holds a fortiori in the contrary case. Estimating $J(S_\varepsilon, rU)$ as in the proof of Theorem 1, we find $J(S_\varepsilon, rU) = o(r^N)$ as $r \to \infty$. Since the number of integer points in an interval differs by no more than 1 from the length of the interval, the estimate $n^*\{(rQ)-\} = r^N m(Q) + o(r^N)$ is valid when Q is one of the constituent cubes of S_ε, hence also for $2\pi^{-1} S_\varepsilon$. Thus from (20),

$$M \geqslant (2\pi)^{-N} r^N m(S_\varepsilon) - o(r^N). \tag{21}$$

Now by Lemma 1 (vi) and (iii)

$$\lambda_k(S_\varepsilon, rU) - \lambda_k(S, rU) \geqslant 0 \quad \text{for all } k,$$

$$\textstyle\sum_k \{\lambda_k(S_\varepsilon, rU) - \lambda_k(S, rU)\} = (2\pi)^{-N} r^N (m(S_\varepsilon) - m(S)) < (2\pi)^{-N} r^N \varepsilon. \tag{22}$$

If M exceeds the index of the eigenvalue of (17), then for every k between the two, the contribution of $\lambda_k(S_\varepsilon, rU) - \lambda_k(S, rU)$ to the left-hand side of (22) exceeds $(1-\gamma)/2 > 0$, hence from (22)

$$M - n(\mathcal{U}+) \leqslant (2\pi)^{-N} r^N \varepsilon 2(1-\gamma)^{-1},$$

an inequality which holds a fortiori in the contrary case. Finally, $n(\mathcal{U}+)$ and $n(\mathcal{U})$ differ by the number of points of Λ lying outside $\mathcal{U}$ but within $d/2$ of $\mathcal{U}$; spheres of radius $d/2$ about each such point form a collection of disjoint spheres, each of fixed volume s_d, all contained in the set of points within d of the boundary of $\mathcal{U}$. Denoting by $V_d(r)$ the volume of the latter set, we see that $V_d(r) = o(r^N)$ and that $V_d(r)/s_d$ serves as an upper bound for $n(\mathcal{U}+) - n(\mathcal{U})$. Thus from (21) and (23)

$$n^+(rU) = n(\mathcal{U}) \geqslant (2\pi)^{-N} r^N \{m(S_\varepsilon) - 2\varepsilon(1-\gamma)^{-1}\} - o(r^N),$$

whence $D^-(\Lambda) \geqslant (2\pi)^{-N}\{m(S_\varepsilon) - 2\varepsilon(1-\gamma)^{-1}\}$, and since $\varepsilon > 0$ is arbitrary, $D^-(\Lambda) \geqslant (2\pi)^{-N} m(S)$. Theorem 3 is established.

THEOREM 4. *Let S be a bounded set in R^N and Λ a set of interpolation for $\mathcal{B}(S)$. Then $D^+(\Lambda) \leqslant (2\pi)^{-N} m(S)$.*

Proof. It is sufficient to prove this theorem for open sets S, since every bounded S may be approximated in measure from the outside by bounded open sets S_n and Λ remains a set of interpolation for $\mathcal{B}(S_n)$. Let $\mathcal{U}$ be a translate of rU such that $n(\mathcal{U}) = n^+(rU) = \max_{\tau \in R^N} n(rU + \tau)$. By Lemma 1 (i) and Lemma 3,

$$\lambda_{n(\mathcal{U}-)-1}(S, rU) = \lambda_{n(\mathcal{U}-)-1}(S, \mathcal{U}) \geqslant \delta > 0,$$

with δ independent of r. Let $J(S, rU) = \sum \lambda_k(S, rU)\{1 - \lambda_k(S, rU)\}$. Given $\eta < 1$, let $M = M(\eta)$ denote the number of eigenvalues $\lambda_k(S, rU)$ which exceed η. If the index of the eigenvalue of (24) exceeds M, then for every k between the two, the contribution to $J(S, rU)$ of the corresponding $\lambda_k(S, rU)$ is at least $\alpha = \min\{\delta(1-\delta), \eta(1-\eta)\} > 0$; thus

$$n(\mathcal{U}-) - M \leqslant \alpha^{-1} J(S, rU), \tag{25}$$

an inequality which holds a fortiori in the contrary case. To estimate $J(S, rU)$ for an open set S, given $\varepsilon > 0$ we choose, as in the proof of Lemma 4, a finite collection S_ε of disjoint scaled translates of U with $S_\varepsilon \subset S$ and $m(S_\varepsilon) > m(S) - \varepsilon$. Evaluating $J(S_\varepsilon, rU)$ as in the proof of Theorem 1, we find $J(S_\varepsilon, rU) = o(r^N)$ as $r \to \infty$. Then by Lemma 1 (iv) and (iii)

$$\begin{aligned} J(S, rU) &= \sum_k \lambda_k(S, rU) - \sum_k \lambda_k^2(S, rU) \leqslant \sum_k \lambda_k(S, rU) - \sum_k \lambda_k^2(S_\varepsilon, rU) \\ &= \sum_k \lambda_k(S, rU) - \sum_k \lambda_k(S_\varepsilon, rU) + J(S_\varepsilon, rU) \\ &= \{m(S) - m(S_\varepsilon)\} r^N + o(r^N) \leqslant \varepsilon r^N + o(r^N), \end{aligned}$$

hence, since ε is arbitrary, $J(S, rU) = o(r^N)$. Now again by Lemma 1 (iii)

$$M \leqslant \sum_k \lambda_k(S, rU) = (2\pi)^{-N} m(S) r^N.$$

Finally, as in the proof of Theorem 3, $n(\mathcal{U}) - n(\mathcal{U}-) = o(r^N)$. Introducing these estimates into (25), we find

$$n^+(rU) = n(\mathcal{U}) \leqslant (2\pi)^{-N} m(S) \eta^{-1} r^N + o(r^N),$$

whence $D^+(\Lambda) \leqslant (2\pi)^{-N} m(S) \eta^{-1}$, and since $\eta < 1$ is arbitrary, $D^+(\Lambda) \leqslant (2\pi)^{-N} m(S)$. Theorem 4 is established.

The conjecture of Beurling

We may now establish Beurling's conjecture regarding the density of sets of balayage. For this section, S will be assumed to be a compact set in R^N, and to satisfy conditions (α) and (β) stated in the introduction. We base ourselves on the following three fundamental theorems, proved by Beurling in the course of his lectures.

THEOREM C. *Let S_d denote the set of all points whose distance to S does not exceed d. If balayage is possible for S and Λ, there exists $d > 0$ such that balayage is possible also for S_d and Λ.*

THEOREM D. *If balayage is possible for S and Λ, there exists a uniformly discrete subset Λ_0 of Λ such that balayage is possible also for S and Λ_0.*

THEOREM E. *If Λ_0 is uniformly discrete and balayage is possible for S and Λ_0, then Λ_0 is a set of sampling for $\mathcal{B}(S)$.*

Since $m(S_d) > m(S)$, the next result is an immediate consequence of Theorem 3.

THEOREM 5. *If balayage is possible for S and Λ, then Λ contains a uniformly discrete subset Λ_0 with $D^-(\Lambda_0) > (2\pi)^{-N} m(S)$.*

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