數學流體力學之理論與計算

Homework Assignment 5

Due date: Prob.1-3 on Dec. 14 2012
Prob.4 on Jan. 04 2013

Part I: Theoretical assignments

Problem 1. Find an orthonormal basis of $L^2(\mathbb{T}^n)$ which is also an orthogonal basis of $H^1(\mathbb{T}^n)$ by looking at the eigenfunctions of $(1-\Delta)$. In other words, define $T : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ by $Tf = u$ if

$$u - \Delta u = f \qquad \text{in} \quad \mathbb{T}^n\,.$$

Show that T is compact, thus by Theorem 4.7 of the Lecture Note one can construct an orthonormal basis of $L^2(\mathbb{T}^n)$ by looking at the eigenvectors of T. State a thoerem similar to Theorem 4.8 of the Lecture Note based on what you see.

Problem 2. Prove Theorem 5.4 of the Lecture Note.

Problem 3. Let Q : $L^2(\mathbb{T}^n)/\mathbb{R} \to H^1(\mathbb{T}^n)$ be defined as in the proof of the Lagrange Multiplier Lemma. Show that Range(Q) is closed. This problem completes the proof of the Lagrange Multiplier Lemma.

Part II: Computational assignments

Problem 4. Consider the Stokes equations on $\mathbb{T}^2$:

$$\begin{aligned} u_t - \Delta u + \nabla p = f \qquad & \text{in} \quad \mathbb{T}^2 \times (0,1]\,, \\ \text{div} u = 0 \qquad & \text{in} \quad \mathbb{T}^2 \times (0,1]\,, \\ u = u_0 \qquad & \text{on} \quad \mathbb{T}^2 \times \{t=0\}\,, \end{aligned}$$

where the initial velocity u_0 and the external forcing f are given by

$$\begin{aligned} u_0(x,y) &= (0,0)\,, \\ f(x,y,t) &= \Big(|y-\pi|\sin\frac{x}{2}, |x-\pi|\cos\frac{y}{2}\Big)\,. \end{aligned}$$

Let N be the number of partitions on each side, and $\Delta t = 0.01$ be the time-step.

1. Use the projection method with non-staggered grid to solve the Stokes equations above numerically, with $N = 25, 50, 100$. Let (u_N, p_N) denote the solution at time $t = 1$. Plot u_N and p_N.

2. Use the penalty method to solve the Stokes equations above numerically, with $N = 25, 50, 100$ and $\theta = 10^{-4}, 10^{-6}$ and 10^{-8}. Let u_N^θ denote the solution at $t = 1$. Plot u_N^θ and $p_N^\theta = -\frac{1}{\theta}\text{div} u_N^\theta$.

3. With the same N, check if the solution u_N^θ converges to u_N as $\theta \to 0$.

Note that you can use the mesh generator of the periodic domain in this problem by re-scaling. You might need the command **sparse** to make the matrix computations more efficient.