Theorem 1.41. *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,*

$$\lim_{x\to a}(f\circ g)(x) = \lim_{x\to a} f(g(x)) = f\Big(\lim_{x\to a} g(x)\Big).$$

Proof. Let $\varepsilon > 0$ be given. Since f is continuous at b, there exists $\delta_1 > 0$ such that

$$|f(x) - f(b)| < \varepsilon \quad \text{whenever} \quad |y - b| < \delta_1 .$$

Fix such a δ_1. Since $\lim_{x\to a} g(x) = b$, there exists $\delta > 0$ such that

$$|g(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - a| < \delta .$$

Therefore, if $0 < |x - a| < \delta$, we must have $|g(x) - b| < \delta_1$ so $|f(g(x)) - f(b)| < \varepsilon$. This shows that $\lim_{x\to a} f(g(x)) = f(b)$. □

Lemma 1.42. *Let x be a real number and $0 \leqslant x < \frac{\pi}{2}$. Then*

$$\sin x \leqslant x \leqslant \tan x . \tag{1.8.1}$$

Moreover,

$$|\sin x| \leqslant |x| \qquad \forall\, x \in \mathbb{R}. \tag{1.8.2}$$

Proof. Inequality (1.8.1) follows from the following figure

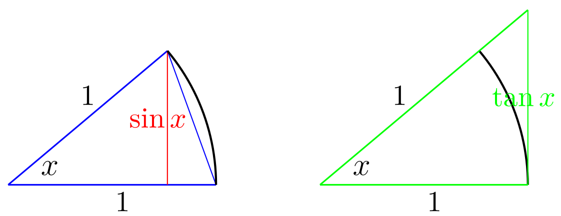

Figure 1.2: The area of the sector is larger than the area of the blue triangle but is smaller than the green triangle.

which shows $\frac{1}{2}\sin x \leqslant \frac{1}{2}x \leqslant \frac{1}{2}\tan x$. This establishes (1.8.1).

To see (1.8.2), by the fact that $|\sin x| \leqslant 1$ it suffice to show that

$$|\sin x| \leqslant |x| \quad \big(\text{or} \quad -|x| \leqslant \sin x \leqslant |x|\big) \qquad \forall\, |x| < \frac{\pi}{2}.$$

Note that by (1.8.1) the inequality above holds for all $x \in \big[0, \frac{\pi}{2}\big)$. The validity of the inequality above for negative x follows from replacing x by $-x$ in the inequality. □

Theorem 1.43. *The sine and cosine function are continuous.*

Proof. This is a direct consequence of the identities

$$\sin x - \sin a = 2\sin\frac{x-a}{2}\cos\frac{x+a}{2} \qquad \text{and} \qquad \cos x - \cos a = 2\sin\frac{x-a}{2}\sin\frac{x+a}{2}$$

and the Squeeze Theorem. □