1.7 The Precise Definition of a Limit

Definition 1.26. Let f be a function defined on some open interval that contains the number a, except possible at a itself. We say that the limit of $f(x)$, as x approaches a, is L, and we write

$$\lim_{x \to a} f(x) = L,$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad \big|f(x) - L\big| < \varepsilon.$$

Remark 1.27. 為了解釋上述定義中關於「對所有⋯存在⋯」句型的意義，我們以一個比較簡單例子看起。「每個不是 2 的質數都是奇數」(Every prime number other than 2 is an odd number) 是一個大家都很清楚的數學描述。但是數學上這個敘述有以下的等價描述：

Every prime number other than 2 is an odd number

$\Leftrightarrow$ Every prime number other than 2 takes the form $2k + 1$ for some integer k

$\Leftrightarrow$ For all/every prime number $p \neq 2$ there exists $k \in \mathbb{Z}$ such that $p = 2k + 1$.

Explanation: 因為 $|f(x) - L| < \varepsilon$ 等價於 $f(x) \in (L - \varepsilon, L + \varepsilon)$，所以定義敘述中的 ε 可視為用來度量 $f(x)$ 向 L 這個數集中的程度。定義所述是指對於任意給定的集中程度 $\varepsilon > 0$，一定可以找到在 c 附近的一個範圍（以到 c 的距離小於 δ 來表示），滿足此範圍中的點之函數值落入想要其落入的集中區域 $(L - \varepsilon, L + \varepsilon)$ 之內。此即「當除 c 之外的點到 c 的距離愈來愈近時，其函數值向 L 集中」的意思。

Example 1.28. In this example we prove, using the definition of limits, that $\lim\limits_{x \to 9} \sqrt{x} = 3$.
Let $\varepsilon > 0$ be given.

1. **The case** $0 < \varepsilon \leqslant 1$: Define $\delta = 6\varepsilon - \varepsilon^2$. Then $\delta > 0$. If $0 < |x - 9| < \delta$, then $9 - \delta < x < 9 + \delta$ which further implies that

$$9 - 6\varepsilon + \varepsilon^2 < x < 9 + 6\varepsilon + \varepsilon^2 \quad \text{or equivalently,} \quad (3 - \varepsilon)^2 < x < (3 + \varepsilon)^2.$$

Therefore, we have $3 - \varepsilon < \sqrt{x} < 3 + \varepsilon$ whenever $0 < |x - 9| < \delta$. Therefore,

$$\text{if } \; 0 < |x - 9| < \delta, \; \text{then } \; \big|\sqrt{x} - 3\big| < \varepsilon.$$

2. **The case** $\varepsilon > 1$: We pick $\delta = 5$ (by letting $\varepsilon = 1$ in the definition of δ given in the previous case). Then the computation above shows that

$$\big|\sqrt{x} - 3\big| < 1 < \varepsilon \qquad \text{whenever} \quad 0 < |x - 9| < \delta.$$

In general, one can show that $\lim\limits_{x \to a} x^{\frac{1}{n}} = a^{\frac{1}{n}}$ if $a > 0$. In order to prove this, for a given $\varepsilon > 0$ what δ you should choose in order to have

$$\big|x^{\frac{1}{n}} - a^{\frac{1}{n}}\big| < \varepsilon \qquad \text{whenever} \quad 0 < |x - a| < \delta?$$

Hint: Prove that if $\delta = \min\left\{\dfrac{a}{2}, \dfrac{na^{\frac{n-1}{n}}\varepsilon}{2}\right\}$, then the statement above holds.

Proof of Part (5) *of Theorem 1.11.* Let $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = K$, and $\varepsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$\big|f(x) - L\big| < \frac{\varepsilon}{2(|K|+1)} \qquad \text{whenever} \quad 0 < |x-a| < \delta_1 .$$

Since $\lim_{x\to a} g(x) = K$, there exists $\delta_2 > 0$ such that

$$\big|g(x) - K\big| < \min\Big\{1, \frac{\varepsilon}{2(|L|+1)}\Big\} \qquad \text{whenever} \quad 0 < |x-a| < \delta_2 .$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$. Moreover, if $0 < |x-a| < \delta$ (which implies that $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ simultaneously),

$$\begin{aligned}\big|f(x)g(x) - LK\big| &= \big|f(x)g(x) - Lg(x) + g(x)L - LK\big| \leqslant \big|g(x)\big|\big|f(x) - L\big| + |L|\big|g(x) - K\big| \\ &= \big|g(x) - K + K\big|\big|f(x) - L\big| + +|L|\big|g(x) - K\big| \\ &\leqslant \big(\big|g(x) - K\big| + |K|\big)\big|f(x) - L\big| + +|L|\big|g(x) - K\big| \\ &\leqslant (1 + |K|)\frac{\varepsilon}{2(|K|+1)} + |L|\frac{\varepsilon}{2(|L|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .\end{aligned}$$

Therefore, we establish that for every given $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < |x-a| < \delta$ then we have $\big|f(x)g(x) - LK\big| < \varepsilon$. This shows that $\lim_{x\to a} \big[f(x)g(x)\big] = LK$. □