1.5 The Limit of a Function

Goal: Given a function f defined near a, we want to know whether the function value $f(x)$ "converges/concentrates" to a certain number if x is "arbitrarily close" to a.

目標：給定一定義在 a 附近的函數 f，我們想知道「當 a 以外的點 x 到 a 的距離愈來愈接近零時，其函數值是否向某數集中」。

Terminologies:

1. **Q**: What does it mean by "a function f defined near a"?

   問：什麼叫做 f 在 a 附近有定義？

   A: It means f is defined on some open interval that contains a except possibly at a.

   答：f 在一包含 a 的開區間上有定義，但在 a 處可能例外沒定義。

2. **Q**: What does it mean by "x is arbitrarily close to a"?

   問：什麼叫做 x 任意靠近 a？

   A: Generally speaking, when we say x is arbitrarily close to a, it means that the distance from x to a is getting closer and closer to zero but not equal to zero. In other words, $|x-a|$ can be made as small as desired but is not zero. When we talk about the function value $f(x)$ being arbitrarily close to a number L, it simply means that $|f(x)-L|$ can be made as small as desired.

   答：一般而言，若是說 x 任意靠近 a，它是指 x 到 a 的距離愈來愈接近零但是非零，也就是說 $|x-a|$ 要多小有多小但是非零；而若是講函數值 $f(x)$ 任意靠近一個數 L，它單純就是指 $|f(x)-L|$ 要多小有多小。

Concerning our goal ／針對我們的目標：

1. If there is such a number (toward which the function value converges), we say that "the limit of $f(x)$, as x approaches a, exists", and denote the number by $\lim\limits_{x\to a} f(x)$.

2. If L is such a number (toward which the function value converges), we say that "the limit of $f(x)$, as x approaches a, equals L" and denote this situation by $\lim\limits_{x\to a} f(x) = L$ or $f(x) \to L$ as $x \to a$.

3. If there is no such a number (toward which the function value converges), we say that "the limit of $f(x)$, as x approaches a, does not exist", and denote this situation by $\lim\limits_{x\to a} f(x)$ D.N.E..

Remark: The name of the independent variable x can be changed to any other symbols, such as t, and $\lim\limits_{x\to a} f(x)$ is the same as $\lim\limits_{t\to a} f(t)$.

Example 1.1. Estimate the value of $\lim\limits_{t\to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	$0.162277\ldots$
± 0.5	$0.165525\ldots$
± 0.1	$0.166620\ldots$
± 0.05	$0.166655\ldots$
± 0.01	$0.166666\ldots$

so we guess $\lim_{t\to 0}\dfrac{\sqrt{t^2+9}-3}{t^2}=\dfrac{1}{6}$.

Example 1.2. Guess the value of $\lim_{x\to 0}\dfrac{\sin x}{x}$.

x	$\dfrac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

so we guess $\lim_{t\to 0}\dfrac{\sin x}{x}=1$ (where x is measured in radian).

Caution: Using the method of substituting values of x that get closer and closer to a and observing whether the function values $f(x)$ converge to a certain number, as shown in the previous two examples, is a poor approach to finding limits. This is especially true when, in many cases, we rely on technological tools to compute function values and can only substitute rational numbers. The fact that we can only observe the function values at rational numbers might lead us to mistakenly believe that the function behaves the same way at irrational numbers, resulting in incorrect conclusions.
如上二例所示的方法，以代入愈來愈接近 a 的 x 並觀察其函數值 $f(x)$ 可能集中到某數去的方式來求極限 $\lim_{x\to a} f(x)$ 是一個糟糕的方法，尤其是在很多時候我們使用科技工具幫忙計算函數值時只能代入有理數，這個只能看到有理數之函數值的事實可能會讓我們誤以為在無理數也有一樣的函數行為，因而導致錯誤的推論。請見下例。

Example 1.3. Find the limit of the Dirichlet function:

$$f(x)=\begin{cases}1 & \text{if } x \text{ is irrational},\\ 0 & \text{if } x \text{ is rational},\end{cases}$$

as x approaches 0. Since $f(\pm 10^{-n})=0$ for all $n\in\mathbb{N}$ (which is like using a calculator or computer to find the function value at ± 0.1, ± 0.01, ± 0.001, etc.), one might expect that $\lim_{x\to 0} f(x)=0$. However, this is incorrect because there are always irrational numbers within any open interval, and f takes the value 1 at irrational numbers. In other words, the function value does not concentrate at 0. Therefore, $\lim_{x\to 0} f(x)$ D.N.E..

1.5.1 One-sided limits

Similar to the case of the limit of a function, we are interested in "whether the function value $f(x)$ converges to a certain number if x is arbitrarily close" to a with x less than a".

1. If there is such a number (toward which the function value converges), we say that "the ***left-hand*** limit of $f(x)$, as x approaches a ***from the left***, exists", and denote the number by $\lim_{x\to a^-} f(x)$.
2. If L is such a number (toward which the function value converges), we say that "the ***left-hand*** limit of $f(x)$, as x approaches a ***from the left***, equals L" and denote this situation by $\lim_{x\to a^-} f(x) = L$ or $f(x) \to L$ as $x \to a^-$.
3. If there is no such a number (toward which the function value converges), we say that "the ***left-hand*** limit of $f(x)$, as x approaches a ***from the left***, does not exist", and denote this situation by $\lim_{x\to a^-} f(x)$ D.N.E..

The right-hand limit $\lim_{x\to a^+} f(x)$ can also be defined in a similar fashion.

Example 1.4. Consider the Heaviside function H defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geqslant 0. \end{cases}$$

Then $\lim_{t\to 0^-} H(t) = 0$ and $\lim_{t\to 0^+} H(t) = 1$.

Theorem 1.5. $\lim_{x\to a} f(x) = L$ *if and only if* $\lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$.

By Theorem 1.5, we conclude that $\lim_{t\to 0} H(t)$ D.N.E..

1.5.2 How can a limit fail to exist?

We have seen from Example 1.3 and 1.4 that the limit of a function might fail to exist due to the fact that the function values of $f(x)$, as x getting closer and closer to a, do not converge/concentrate at one ***single*** number. Here comes another important example.

Example 1.6. Investigate $\lim_{x\to 0} \sin\frac{\pi}{x}$. (D.N.E.)

Another possibility that the limit fail to exist:

Example 1.7. Find $\lim_{x\to 0} \frac{1}{x^2}$. (D.N.E.)

We use the notation $\lim_{x\to a} f(x) = \infty$ to indicate that "the value of $f(x)$ can be made arbitrarily large (as large as we want) by taking x sufficiently close to a but not equal to a". It can also be written as $f(x) \to \infty$ as $x \to a$, and read as

1. the limit of $fx)$, as x approaches a, is infinity; or

2. $f(x)$ increases without bound as x approaches a.

Definition 1.8. The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least on of the following statements is true:

1. $\lim_{x\to a} f(x) = \infty$; 2. $\lim_{x\to a^-} f(x) = \infty$; 3. $\lim_{x\to a^+} f(x) = \infty$;

4. $\lim_{x\to a} f(x) = -\infty$; 5. $\lim_{x\to a^-} f(x) = -\infty$; 6. $\lim_{x\to a^+} f(x) = -\infty$.

Example 1.9. The curve $y = \dfrac{2x}{x-3}$ has one (and only one) vertical asymptote $x = 3$ since $\lim_{x\to 3^+} \dfrac{2x}{x-3} = \infty$.

Example 1.10. The vertical asymptotes of the curve $y = \tan x$ are $x = n\pi + \dfrac{\pi}{2}$, $n \in \mathbb{Z}$.

1.6 Calculating Limits Using the Limit Laws

Theorem 1.11. *Let c be a constant.*

1. $\lim_{x\to a} c = c$. 2. $\lim_{x\to a} x = a$.

Theorem 1.12. *Suppose that c is a constant and the limits*

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} \big[f(x) + g(x)\big] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. **(Sum Law)**

2. $\lim_{x\to a} \big[f(x) - g(x)\big] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$. **(Difference Law)**

3. $\lim_{x\to a} \big[f(x)g(x)\big] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$. **(Product Law)**

4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$. **(Quotient Law)**

Using the product law, the **Constant Multiple Law**

$$\lim_{x\to a} \big[cf(x)\big] = c \lim_{x\to a} f(x)$$

and the **Power Law**

$$\lim_{x\to a} \big[f(x)\big]^n = \Big[\lim_{x\to a} f(x)\Big]^n \qquad \text{for all } n \in \mathbb{N}.$$

can be obtained.

Theorem 1.13 (Root Law)**.** *Let n be a positive integer.*

1. *If n is odd, then* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$.

2. *If n is even and* $\lim_{x\to a} f(x) > 0$, *then* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$.

Theorem 1.14 (Direct Substitution Property)**.** *If f is a polynomial or a rational function and a is in the domain of f, then* $\lim_{x\to a} f(x) = f(a)$.

Theorem 1.15. *Let f, g be functions defined on an open interval containing a (except possibly at c), and $f(x) = g(x)$ if $x \neq a$. If* $\lim_{x\to a} g(x) = L$, *then* $\lim_{x\to a} f(x) = L$.