Extra Exercise Problem Sets 3

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Problem 1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that f has only one critical point $c \in (a, b)$.

1. Show that if $f(c)$ is a local extremum of f, then $f(c)$ is an absolute extremum of f.
2. Show that if $f(c)$ is the absolute minimum of f, then $f(x) > f(c)$ for all $x \in [a, b]$ and $x \neq c$. Similarly, show that if $f(c)$ is the absolute maximum of f, then $f(x) < f(c)$ for all $x \in [a, b]$ and $x \neq c$.

Problem 2. Let I, J be intervals, $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be increasing functions. Show that if J contains the range of g, then $f \circ g$ is increasing on I.

Problem 3. 1. If the function $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-\dfrac{2\sqrt{3}}{9}$ at $x = \dfrac{1}{\sqrt{3}}$, what are the values of a and b?

2. Which of the tangent lines to the curve in part (1) has the smallest slope?

Problem 4. A number a is called a fixed point of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

Problem 5. Suppose f is an odd function (that is, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$) and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = \dfrac{f(b)}{b}$.

Problem 6. Show that $2\sqrt{x} > 3 - \dfrac{1}{x}$ for all $x > 1$.

Problem 7. Show that $\sqrt{b} - \sqrt{a} < \dfrac{b-a}{2\sqrt{a}}$ for all $0 < a < b$.

Problem 8. Show that for all (rational numbers) $p, q \in (1, \infty)$ satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have

$$ac + bd \leqslant (a^p + b^p)^{\frac{1}{p}} (c^q + d^q)^{\frac{1}{q}} \qquad \forall\, a, b, c, d > 0\,.$$

Hint: Let $x = \dfrac{a}{b}$ and $y = \dfrac{d}{c}$.

Problem 9. Show that for all $k \in \mathbb{N} \cup \{0\}$,

$$x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} \leqslant \sin x \leqslant x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} \qquad \forall\, x \geqslant 0\,,$$

$$1 - \frac{x^2}{2!} + \cdots + \frac{x^{4k}}{(4k)!} - \frac{x^{4k+2}}{(4k+2)!} \leqslant \cos x \leqslant 1 - \frac{x^2}{2} + \cdots + \frac{x^{4k}}{(4k)!} \qquad \forall\, x \geqslant 0\,.$$

Problem 10. （不要用交叉相乘）Show that for all $k \in \mathbb{N} \cup \{0\}$,

$$1 - x + x^2 - x^3 + \cdots + x^{2k} - x^{2k+1} \leqslant \frac{1}{1+x} \leqslant 1 - x + x^2 - x^3 + \cdots + x^{2k} \qquad \forall\, x \geqslant 0\,.$$

Problem 11. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying that $f'(x) = f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$.

1. （不要試著找出 f 而是直接用 f 的性質）Show that f is increasing on $\mathbb{R}$.
2. Show that if $k \in \mathbb{N} \cup \{0\}$, then $f(x) \geqslant 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for all $x \geqslant 0$.
3. Show that if $k \in \mathbb{N} \cup \{0\}$, then

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} + \frac{x^{2k+1}}{(2k+1)!} \leqslant f(x) \leqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} \qquad \forall\, x \leqslant 0\,.$$

Hint: 1. Show that f^2 is increasing on $\mathbb{R}$ and argue that f is also increasing on $\mathbb{R}$.

Problem 12. Find the minimum value of

$$\big|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x\big|$$

for real numbers x.
Hint: Let $t = \sin x + \cos x$.

Problem 13. Let $f, g : (a, b) \to \mathbb{R}$ be twice differentiable functions such that $f''(x) \neq 0$ and $g''(x) \neq 0$ for all $x \in (a, b)$. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).

Problem 14. For what values of a and b is $(2, 2.5)$ an inflection point of the curve $x^2 + ax + by = 0$? What additional inflection points does the curve have?

Problem 15. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b \cos x\, dx$ by the following steps.

(a) Partition $[a, b]$ into n sub-intervals with equal length. Write down the Riemann sum using the right end-point rule.

(b) Prove that

$$\sum_{i=1}^{n} \cos(a + id) = \frac{\sin\left[a + \left(n + \frac{1}{2}\right)d\right] - \sin\left(a + \frac{d}{2}\right)}{2\sin\dfrac{d}{2}}\,. \qquad (\star)$$

Hint: Use the sum and difference formula $\sin(\vartheta + \varphi) - \sin(\vartheta - \varphi) = 2\sin\vartheta\cos\varphi$.

(c) Use $(\star)$ to simplify the Riemann sum in (a), and find the limit of the Riemann sum as n approaches infinity. Show that

$$\int_a^b \cos x\, dx = \sin b - \sin a\,.$$

Problem 16. Let $a < b$ be real numbers. Compute $\int_a^b x^N \, dx$, where N is a non-negative integer, by the following steps.

(a) Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a regular partition of $[a, b]$. Show that the Riemann sum using the right end-point rule is given by

$$I_n = \sum_{k=0}^{N} \left[C_k^N a^{N-k} (b-a)^{k+1} \Big(\frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k \Big) \right],$$

where $C_k^N = \dfrac{N!}{k!(N-k)!}$.

(b) Show that

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1}(n+1)^{k+1} - \frac{1}{k+1}\left[C_{k-1}^{k+1} \sum_{i=1}^{n} i^{k-1} + \cdots + C_1^{k+1} \sum_{i=1}^{n} i + (n+1) \right]. \qquad (\star\star)$$

Hint: Expand $(j+1)^k$ for $j = 0, 1, 2, \cdots, n$ by the binomial expansion formula, and sum over j to obtain the equality above.

(c) Use $(\star\star)$ to show that $\displaystyle\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k = \frac{1}{k+1}$ for each $k \in \mathbb{N}$.

(d) Use the limit in (c) to find the limit of the Riemann sum in (a) by passing to the limit as n approaches infinity. Simplify the result to show that

$$\int_a^b x^N \, dx = \frac{b^{N+1} - a^{N+1}}{N+1}.$$

Hint: (c) By induction!

Problem 17. Use the limit of Riemann sums to compute the integral $\int_0^\pi x \cos x \, dx$.

Problem 18. Find the following definite integrals (without using any further techniques of integrations).

1. $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx.$
2. $\displaystyle\int_0^{\frac{\pi}{3}} (\cos x + \sec x)^2 \, dx.$
3. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x} \, dx.$
4. $\displaystyle\int_0^{\frac{\pi}{6}} (\sec x + \tan x)^2 \, dx.$
5. $\displaystyle\int_0^{\pi} (\cos x + |\cos x|) \, dx.$
6. $\displaystyle\int_0^{\frac{3\pi}{2}} |\sin x| \, dx.$
7. $\displaystyle\int_0^{4} |x^2 - 4x + 3| \, dx.$

Problem 19. Find the following derivatives.

1. $\displaystyle\frac{d}{dx} \int_0^{\sqrt{x}} \cos t \, dt.$
2. $\displaystyle\frac{d}{dx} \int_1^{\sin x} 3t^2 \, dt.$
3. $\displaystyle\frac{d}{dx} \int_0^{\tan x} \sec^2 t \, dt.$
4. $\displaystyle\frac{d}{dx} \int_0^{\sqrt{x}} \Big(t^4 + \frac{3}{\sqrt{1-t^2}} \Big) \, dt.$
5. $\displaystyle\frac{d}{dx} \int_2^{x^2} \sin(t^3) \, dt.$
6. $\displaystyle\frac{d}{dx} \int_0^{\sin x} \frac{1}{\sqrt{1-t^2}} \, dt, \ |x| < \frac{\pi}{2}.$

7. $\dfrac{d}{dx}\displaystyle\int_0^{\tan x} \frac{1}{1+t^2}\,dt.$

Problem 20. Find an anti-derivative of the function $f(x) = |x|$.

Problem 21. Let $G(x) = \displaystyle\int_0^x \left[s \int_0^s f(t)\,dt \right] ds$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Find

1. $G(0)$. 2. $G'(0)$. 3. $G''(0)$. 4. $G''(x)$.

Problem 22. Suppose that f has a positive derivative for all values of x (that is, $f'(x) > 0$ for all $x \in \mathbb{R}$) and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a. g is a differentiable function of x.

b. g is a continuous function of x.

c. The graph of g has a horizontal tangent at $x = 1$.

d. g has a local maximum at $x = 1$.

e. g has a local minimum at $x = 1$.

f. The graph of g has an inflection point at $x = 1$.

g. The graph of $\dfrac{dg}{dx}$ crosses the x-axis at $x = 1$.

Problem 23. For each continuous function $f : [0, 1] \to \mathbb{R}$, let

$$I(f) = \int_0^1 x^2 f(x)\,dx \qquad \text{and} \qquad J(f) = \int_0^1 x f(x)^2\,dx\,.$$

Find the maximum value of $I(f) - J(f)$ over all such functions f.