Extra Exercise Problem Sets 2

Oct. 13. 2024

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy $f\big(\frac{x^2-1}{x^2+1}\big) = x$ for all $x > 0$. Find $f'(0)$.

Problem 2. 1. Let $n \in \mathbb{N}$. Show that $\frac{d}{dx}\big[\sin^n x \cos(nx)\big] = n \sin^{n-1} x \cos(n+1)x$.

2. Find a similar formula for the derivative of $\cos^n x \cos(nx)$.

Problem 3. Find the derivative of the following functions:

1. $y = \cos\sqrt{\sin(\tan(\pi x))}$.
2. $y = \big[x + (x + \sin^2 x)^3\big]^4$.

Problem 4. The graph of the functions "arcsin" and "arccos" are the blue and green part of the curve consisting of points (x, y) satisfying $\sin y = x$ and $\cos y = x$, respectively, given below

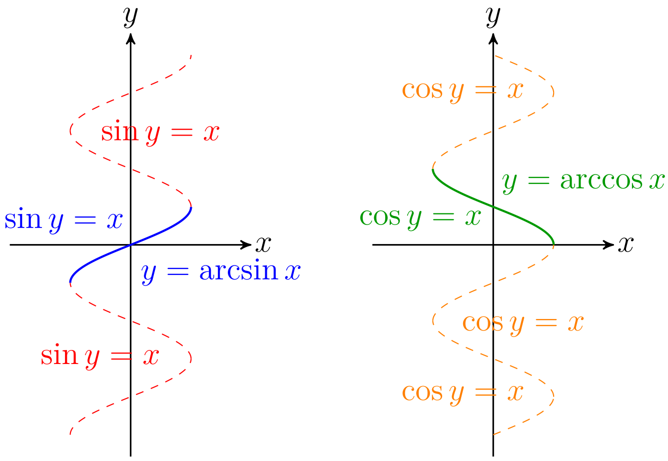

Figure 1: The graph of functions $y = \arcsin x$ and $y = \arccos x$

1. Find the domain and the range of the two functions arcsin and arccos.
2. Show that $\sin(\arcsin x) = x$ for all x in the domain of arcsin and $\cos(\arccos x) = x$ whenever x in the domain of arccos.
3. Is it true that $\arcsin(\sin x) = x$ or $\arccos(\cos x) = x$?
4. Find $\sin(\arccos x)$ and $\cos(\arcsin x)$.
5. Show that $\frac{d}{dx}\Big|_{x=c}\big(\arcsin x + \arccos x\big) = 0$ for all c in both domains.
6. Find $\frac{d}{dx}\arcsin\frac{1}{x}$ and $\frac{d}{dx}(\arccos x)^2$.

Problem 5. The function arctan is defined similarly to functions arcsin and arccos: consider the collection of all points (x, y) satisfying $\tan y = x$ (see the figure below), and the blue part is the graph of a function called "arctan".

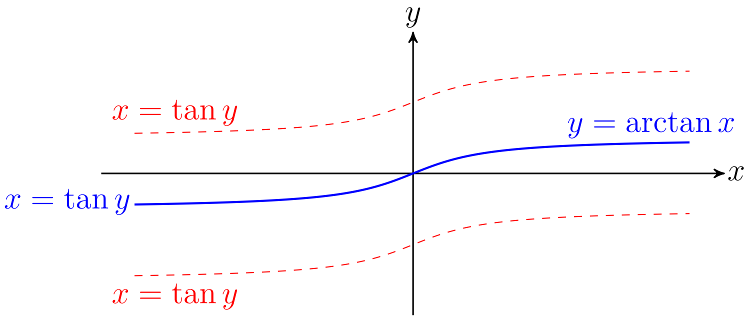

Figure 2: The graph of function $y = \arctan x$

1. Find the domain and the range of the function arctan.

2. Show that $\tan(\arctan x) = x$ for all x in the domain of arctan.

3. Is is true that $\arctan(\tan x) = x$ for all x in the domain of tan?

4. Find $\dfrac{d}{dx} \arctan x$.

Problem 6. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if $\sin(x+y) = y^2 \cos x$.

Problem 7. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

Problem 8. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y} = \sqrt{c}$ is equal to c.

Problem 9. The Bessel function of order 0, denoted by $y = J_0(x)$, satisfies the differential equation

$$xy'' + y' + xy = 0$$

for all values of x and its value at 0 is $J_0(0) = 1$.

1. Find $J_0'(0)$.

2. Use implicit differentiation to find $J_0''(0)$.