Chapter 15

Vector Analysis

15.1 Vector Fields

In Chapter 12 and 13 we have talked about vector-valued functions of one variable and real-valued functions (also called ***scalar*** functions) of several variables, respectively. In this chapter, we will focus on the properties a special type of vector-valued functions of several variables (which is a correspondence which assigns to each $(x_1, \cdots, x_n)$ in a certain region a vector $(y_1, \cdots, y_n)$ in $\mathbb{R}^n$).

Definition 15.1

A (two-dimensional) vector field over a plane region R is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x, y) \in \mathbb{R}^2$ to each point (x, y) in R. A (three-dimensional) vector field over a solid region Q is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x, y, z) \in \mathbb{R}^3$ to each point (x, y, z) in Q.

In general, an n-dimensional vector field over a region $D \subseteq \mathbb{R}^n$ is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ to each point $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ in D.

Definition 15.2: Conservative Vector Fields

A vector field $\boldsymbol{F}$ is called conservative if there exists a differentiable scalar function ϕ such that $\boldsymbol{F} = \nabla\phi$. In such a case, the function ϕ is called the ***potential function*** for $\boldsymbol{F}$.

Example 15.3. Let M and m be the mass of the earth and a satellite. Introduce a Cartesian coordinate system whose origin is the center of mass of the earth. If the position of the satellite is $\boldsymbol{x}$, then the gravity force acting on the satellite is

$$\boldsymbol{F}(\boldsymbol{x}) = -\frac{GMm}{\|\boldsymbol{x}\|^2}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = -\frac{GMm}{\|\boldsymbol{x}\|^3}\boldsymbol{x}$$

which is a conservative vector field since $\phi(\boldsymbol{x}) = \dfrac{GMm}{\|\boldsymbol{x}\|}$ is a potential function for $\boldsymbol{F}$.

We remark that when moving an object with mass m located at height h_1 to some place with height h_2, the potential of that object differs by the amount

$$\frac{GMm}{R+h_2} - \frac{GMm}{R+h_1} = m\frac{GM}{R}\Big(\frac{1}{1+h_2/R} - \frac{1}{1+h_1/R}\Big) \overset{(h_1,h_2 \ll R)}{\approx} m\frac{GM}{R}\Big(\frac{h_1}{R} - \frac{h_2}{R}\Big) = mg(h_1-h_2)\,,$$

where $g = \dfrac{GM}{R^2}$ is the gravitational acceleration; thus we recover what we learn from high school physics (but this is in fact an approximation).

Example 15.4. Find the potential function for the conservative vector field

$$\boldsymbol{F}(x,y,z) = 2xy\mathbf{i} + (x^2+z^2)\mathbf{j} + 2yz\mathbf{k}\,.$$

Suppose that $\boldsymbol{F} = \nabla\phi$. Then $\phi_x(x,y,z) = 2xy$ which implies that

$$\phi(x,y,z) = x^2y + f(y,z)$$

for some function f. Comparing the second component, we find that

$$x^2+z^2 = \phi_y(x,y,z) = x^2 + f_y(y,z)\,;$$

thus $f_y(y,z) = z^2$ or $f(y,z) = yz^2 + g(z)$. Finally, to determine g we compare the third component and obtain that

$$2yz = \phi_z(x,y,z) = f_z(y,z) = 2yz + g'(z)\,;$$

thus $g(z) = K$ for some constant K. Therefore, $\phi(x,y,z) = x^2y + yz^2 + K$.

Definition 15.5: 旋度

Let Q be an open region in space, and $\boldsymbol{F} : Q \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$. The curl of $\boldsymbol{F}$, also called the vorticity of $\boldsymbol{F}$, is a vector field given by

$$\text{curl}\boldsymbol{F} = \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\mathbf{i} - \Big(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k}\,.$$

If $\text{curl}\boldsymbol{F} = \mathbf{0}$, then $\boldsymbol{F}$ is said to be ***irrotational***.

Symbolically, the curl of $\boldsymbol{F}$ is given by

$$\mathrm{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} .$$

Remark 15.6. Let $\boldsymbol{F}$ be a three-dimensional vector field, and F_i be the i-th component of $\boldsymbol{F}$; that is,

$$\boldsymbol{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \sum_{i=1}^{3} F_i\mathbf{e}_i .$$

Then using the permutation symbol ε_{ijk}, we have

$$(\mathrm{curl}\boldsymbol{F})_i \equiv \text{the } i\text{-th component of } \mathrm{curl}\boldsymbol{F} = \sum_{j,k=1}^{3} \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j} . \tag{15.1.1}$$

Remark 15.7. Let $\boldsymbol{F}$ be a two dimensional vector field given by $\boldsymbol{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{k}$. We can also define the curl of $\boldsymbol{F}$ by treating $\boldsymbol{F}$ as a three-dimensional vector field

$$\widetilde{\boldsymbol{F}}(x,y,z) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} + 0\mathbf{k}$$

(which is a three-dimensional vector field independent of z) and define curl$\boldsymbol{F}$ as the third component of curl$\widetilde{\boldsymbol{F}}$ (for the first two components of curl$\widetilde{\boldsymbol{F}}$ are zero). Therefore, the curl of a two dimensional vector field $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$ is a scalar function given by

$$\mathrm{curl}\boldsymbol{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} .$$

Moreover, by defining the differential operator $\nabla^\perp = \Big(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\Big)$ on plane we have the symbolic representation

$$\mathrm{curl}\boldsymbol{F} = \nabla^\perp \cdot \boldsymbol{F} .$$

Theorem 15.8

If $\boldsymbol{F}$ is a continuously differentiable conservative three-dimensional vector field (which means each component of $\boldsymbol{F}$ has continuous first partial derivative), then curl$\boldsymbol{F} = \mathbf{0}$.

Proof. Suppose that $\boldsymbol{F} = (F_1, F_2, F_3) = \nabla\phi$ for some differentiable potential function ϕ. Then $F_k = \dfrac{\partial \phi}{\partial x_k}$. Moreover, since $\boldsymbol{F}$ is continuously differentiable, ϕ_{x_1}, ϕ_{x_2} and ϕ_{x_3} are continuously differentiable so that ϕ has continuous mixed second partial derivatives. Therefore,

$\dfrac{\partial^2 \phi}{\partial x_j \partial x_k} = \dfrac{\partial^2 \phi}{\partial x_k \partial x_j}$ which further implies that

$$(\text{curl}\boldsymbol{F})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} = \sum_{j,k=1}^{3} \varepsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} = \sum_{j,k=1}^{3} \varepsilon_{ijk} \frac{\partial^2 \phi}{\partial x_k \partial x_j} = \sum_{j,k=1}^{3} \varepsilon_{ikj} \frac{\partial^2 \phi}{\partial x_j \partial x_k} ,$$

where the last equality is obtained by switching k, j indices in the former expression. Therefore, by the fact that $\varepsilon_{ikj} = -\varepsilon_{ijk}$, we conclude that $(\text{curl}\boldsymbol{F})_i = 0$ for all $1 \leqslant i \leqslant 3$. □

Definition 15.9: 散度

Let R be an open region in the plane, and $\boldsymbol{F} : R \to \mathbb{R}^2$ be a vector field given by $\boldsymbol{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. The divergence of $\boldsymbol{F}$ is a scalar function given by

$$\text{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} .$$

Let Q be an open region in space, and $\boldsymbol{F} : Q \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. The divergence of $\boldsymbol{F}$ is a scalar function given by

$$\text{curl}\boldsymbol{F} = \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\mathbf{i} - \Big(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k} .$$

In general, if D is an open region in $\mathbb{R}^n$ and $\boldsymbol{F} : D \to \mathbb{R}^n$ be a vector field given by $\boldsymbol{F}(\boldsymbol{x}) = \big(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \cdots, F_n(\boldsymbol{x})\big)$, the divergence of $\boldsymbol{F}$ is a scalar function given by

$$\text{div}\boldsymbol{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} .$$

Theorem 15.10

Let $\boldsymbol{F}$ be a three-dimensional vector field given by $\boldsymbol{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. If M, N, P have continuous second partial derivatives, then

$$\text{div}(\text{curl}\boldsymbol{F}) = 0 .$$

Proof. Let $F_1 = M$, $F_2 = N$ and $F_3 = P$ so that we can write $\boldsymbol{F} = \sum\limits_{i=1}^{3} F_i \mathbf{e}_i$. Then the fact that M, N, P have continuous second partial derivatives implies that $\dfrac{\partial^2 F_k}{\partial x_i \partial x_j} = \dfrac{\partial^2 F_k}{\partial x_j \partial x_i}$ for each $1 \leqslant k \leqslant 3$; thus

$$\begin{aligned}
\operatorname{div}(\operatorname{curl}\boldsymbol{F}) &= \sum_{i=1}^{3} \frac{\partial(\operatorname{curl}\boldsymbol{F})_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(\sum_{j,k=1}^{3} \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j}\Big) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\frac{\partial^2 F_k}{\partial x_i \partial x_j} \\
&= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\frac{\partial^2 F_k}{\partial x_j \partial x_i} = \sum_{i,j,k=1}^{3} \varepsilon_{jik}\frac{\partial^2 F_k}{\partial x_i \partial x_j}\,,
\end{aligned}$$

where the last equality is obtained by switching k, j indices in the former expression. Therefore, by the fact that $\varepsilon_{jik} = -\varepsilon_{ijk}$, we conclude that $\operatorname{div}(\operatorname{curl}\boldsymbol{F}) = 0$. $\square$

15.2 Line Integrals

15.2.1 Line integrals of scalar functions

In this section, we are concerned with the "integral" of a real-valued function f defined on a curve C. It is motivated by calculating the total mass of a wire lying along a curve in space when the density of the wire at each point is given, or to find the work done by a given (variable) force acting along such a curve. We begin with the following

Definition 15.11

Let C be a curve in space. A ***partition*** of C is a collection of curves $\big\{C_1, C_2, \cdots, C_n\big\}$ satisfying

1. $C = \bigcup\limits_{i=1}^{n} C_i$ (so that $C_i \subseteq C$);
2. If $i \neq j$, then $C_i \cap C_j$ contains at most two points.

Let $\mathcal{P} = \big\{C_1, C_2, \cdots, C_n\big\}$ be a partition of C. The ***norm*** of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the number

$$\|\mathcal{P}\| = \max\big\{\ell(C_1), \ell(C_2), \cdots, \ell(C_n)\big\}\,,$$

where $\ell(C_j)$ denotes the length of curve C_j. If $f : C \to \mathbb{R}$ is a real-valued function defined on C, a ***Riemann sum*** of f for partition $\mathcal{P}$ is a sum of the form

$$\sum_{i=1}^{n} f(q_i)\ell(C_i)\,,$$

where $\{q_1, q_2, \cdots, q_n\}$ is a collection of points on C satisfying $q_j \in C_j$ for all $1 \leqslant j \leqslant n$.

We note that **in order to define the norm of partitions, it is required that every sub-curve C_j of C has length**. This kind of curves is called ***rectifiable*** curves, and we can only consider line integrals along rectifiable curves.

Similar to the Riemann integral, we consider the limit of Riemann sums

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} f(q_j)\ell(C_j)\,.$$

The line integral of f along C is the limit above if the limit indeed exists. The precise definition of the limit is similar to those given in Definition 4.7, 14.1 and 14.28 and is given below.

Definition 15.12

Let C be a rectifiable curve, and $f : C \to \mathbb{R}$ be a scalar function. The line integral of f along C is a real number L such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{P} = \{C_1, C_2, \cdots, C_n\}$ is a partition of C satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to the interval $(L - \varepsilon, L + \varepsilon)$.

Whenever such an L exists, it must be unique, and the number L is denoted by $\displaystyle\int_C f\,ds$ $\big($and when C is a closed curve, we use $\displaystyle\oint_C f\,ds$ to emphasize that the curve is closed$\big)$.

To discuss the existence of line integrals, we first define the continuity of f on C.

Definition 15.13

Let C be a curve, and $f : C \to \mathbb{R}$ be a scalar function. f is said to be continuous at a point $p \in C$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f(q) - f(p)\| < \varepsilon \quad \text{whenever} \quad \|p - q\| < \delta \text{ and } q \in C\,.$$

If f is continuous at every point of C, then f is said to be continuous on C.

Remark 15.14. Let C be a curve. If f is continuous in an open region containing C, then f is continuous on C.

Similar to Theorem 4.10 and 4.17, we have the following two theorems.

Theorem 15.15

Let C be a rectifiable curve, and $f : C \to \mathbb{R}$ be a bounded continuous function. Then the line integral of f along C exists.

Theorem 15.16

Let C be a rectifiable curve, and $f : C \to \mathbb{R}$ be a real-valued function such that the line integral of f along C exists. If $\mathcal{P} = \big\{C_1, C_2, \cdots, C_n\big\}$ is a partition of C, then

$$\int_C f\, ds = \sum_{j=1}^{n} \int_{C_i} f\, ds\,.$$

To compute the length of curves, we rely on Theorem 12.35; thus in the following we assume that the curve C under consideration has an injective continuous differentiable parametrization $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$. If $\mathcal{P} = \{C_1, C_2, \cdots, C_n\}$ is a partition of C, there exists a partition $\mathcal{Q} = \big\{a = t_0 < t_1 < \cdots < t_n = b\big\}$ of $[a, b]$ such that for each $1 \leqslant j \leqslant n$, $C_j = \boldsymbol{r}([t_{k-1}, t_k])$ for some $1 \leqslant k \leqslant n$. By relabelling the curves in the partition, W.L.O.G. we can always assume that $C_j = \boldsymbol{r}([t_{j-1}, t_j])$ for all $1 \leqslant j \leqslant n$ so that we can alternatively define partitions of a curve C by

A collection of curves $\mathcal{P} = \big\{C_1, C_2, \cdots, C_n\big\}$ is a partition of curve C if there exists a partition $\mathcal{Q} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ satisfying that $C_j = \boldsymbol{r}([t_{j-1}, t_j])$.

Moreover, Theorem 12.35 implies that

$$\ell(C_j) = \int_{t_{j-1}}^{t_j} \|\boldsymbol{r}'(t)\|\, dt = \big\|\boldsymbol{r}'(d_j)\big\|(t_j - t_{j-1})\,, \tag{15.2.1}$$

where the mean value theorem for integrals is used to obtain the last equality. We also observe that by assuming that $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in [a, b]$,

$$\|\mathcal{P}\| \to 0 \;\text{ if and only if }\; \|\mathcal{Q}\| \to 0\,. \tag{15.2.2}$$

Suppose that f be a bounded scalar function defined on C (so that one can think of the function $f \circ \boldsymbol{r} : [a, b] \to \mathbb{R}$), and $\sum\limits_{j=1}^{n} f(q_j)\ell(C_j)$ is a Riemann sum of f for partition $\mathcal{P}$. For each $1 \leqslant j \leqslant n$, $q_j \in C_j = \boldsymbol{r}([t_{j-1}, t_j])$; thus there exists

$$q_j = \boldsymbol{r}(c_j) \quad \text{for some } c_j \in [t_{j-1}, t_j]\,. \tag{15.2.3}$$

Therefore, with the help of (15.2.1) and (15.2.3),

$$\sum_{j=1}^{n} f(q_j)\ell(C_j) = \sum_{j=1}^{n} (f \circ \boldsymbol{r})(c_j)\big\|\boldsymbol{r}'(d_j)\big\|(t_j - t_{j-1})\,.$$

Using (15.2.2), we find that

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} f(q_j)\ell(C_j) = \lim_{\|\mathcal{Q}\|\to 0} \sum_{j=1}^{n} (f \circ \boldsymbol{r})(c_j)\big\|\boldsymbol{r}'(d_j)\big\|(t_j - t_{j-1})\,.$$

Since

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{i=1}^{n} (f \circ \boldsymbol{r})(c_i)\|\boldsymbol{r}'(c_i)\|(t_i - t_{i-1}) - \lim_{\|\mathcal{P}\|\to 0} \sum_{i=1}^{n} (f \circ \boldsymbol{r})(c_i)\|\boldsymbol{r}'(d_i)\|(t_i - t_{i-1}) = 0\,,$$

(here some rigorous argument involving the uniform continuity of $\|\boldsymbol{r}'\|$ has to be made here),

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} f(q_j)\ell(C_j) = \lim_{\|\mathcal{Q}\|\to 0} \sum_{j=1}^{n} (f \circ \boldsymbol{r})(c_j)\big\|\boldsymbol{r}'(c_j)\big\|(t_j - t_{j-1})\,.$$

We note that the sum in the right-hand side is the Riemann sum of a function g for partition $\mathcal{Q}$ given by $g(t) = (f \circ \boldsymbol{r})(t)\|\boldsymbol{r}'(t)\|$.

Theorem 15.17

Let C be a (piecewise) smooth curve with (piecewise) continuously differentiable injective parametrization $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$, and $f : C \to \mathbb{R}$ be a continuous function. Then the line integral of f along C exists and is given by

$$\int_a^b (f \circ \boldsymbol{r})(t)\big\|\boldsymbol{r}'(t)\big\|\,dt\,.$$

Example 15.18. Evaluate $\displaystyle\int_C (x^2 - y + 3z)\,ds$, where C is the line segment connecting the points $(0, 0, 0)$ and $(1, 2, 1)$.

First we note that the line segment can be parameterized by

$$\boldsymbol{r}(t) = (1 - t)(0, 0, 0) + t(1, 2, 1) = (t, 2t, t) \qquad t \in [0, 1]\,.$$

Therefore, Theorem 15.17 implies that

$$\int_C (x^2 - y + 3z)\,ds = \int_0^1 (t^2 - 2t + 3t)\|(1, 2, 1)\|\,dt = \sqrt{6}\int_0^1 (t^2 + t)\,dt = \frac{5\sqrt{6}}{6}\,.$$

Example 15.19. Evaluate $\displaystyle\int_C x\,ds$, where C is the piecewise smooth curve starting from $(0,0)$ to $(1,1)$ along $y = x^2$ then from $(1,1)$ to $(0,0)$ along $y = x$.

Let C_1 be the piece of the curve connecting $(0,0)$ and $(1,1)$ along $y = x^2$, and C_2 be the piece of the curve connecting $(1,1)$ and $(0,0)$ along $y = x$. Then C_1 and C_2 can be parameterized by

$$\boldsymbol{r}_1(t) = (t, t^2) \quad t \in [0,1] \qquad \text{and} \qquad \boldsymbol{r}_2(t) = (t,t) \quad t \in [0,1]\,,$$

respectively. Since $C = C_1 \cup C_2$ and $C_1 \cap C_2$ has only two points,

$$\begin{aligned}\int_C x\,ds &= \int_{C_1} x\,ds + \int_{C_2} x\,ds = \int_0^1 t\|(1,2t)\|\,dt + \int_0^1 t\|(1,1)\|\,dt = \int_0^1 \big[t\sqrt{1+4t^2} + \sqrt{2}t\big]\,dt \\ &= \Big[\frac{1}{12}(1+4t^2)^{\frac{3}{2}} + \frac{\sqrt{2}t^2}{2}\Big]\Big|_{t=0}^{t=1} = \frac{1}{12}(5\sqrt{5}-1) + \frac{\sqrt{2}}{2}\,.\end{aligned}$$

Example 15.20. Let C be the upper half part of the circle centered at the origin with radius $R > 0$ in the xy-plane. Evaluate the line integral $\displaystyle\int_C y\,ds$.

First, we parameterize C by

$$\boldsymbol{r}(t) = (R\cos t, R\sin t) \qquad t \in [0,\pi]\,.$$

Then

$$\int_C y\,ds = \int_0^\pi R\sin t\big\|(-R\sin t, R\cos t)\big\|_{\mathbb{R}^2} dt = \int_0^\pi R^2 \sin t\,dt = 2R^2\,.$$

Example 15.21. Find the mass of a wire lying along the first octant part of the curve of intersection of the elliptic paraboloid $z = 2 - x^2 - 2y^2$ and the parabolic cylinder $z = x^2$ between $(0,1,0)$ and $(1,0,1)$ if the density of the wire at position (x,y,z) is $\varrho(x,y,z) = xy$.

Note that we can parameterize the curve C by

$$\boldsymbol{r}(t) = (t, \sqrt{1-t^2}, t^2) \qquad t \in [0,1]\,.$$

Therefore, the mass of the curve can be computed by

$$\begin{aligned}\int_C \varrho\,ds &= \int_0^1 t\sqrt{1-t^2}\,\big\|(1, \frac{-t}{\sqrt{1-t^2}}, 2t)\big\|_{\mathbb{R}^3} dt = \int_0^1 t\sqrt{1-t^2}\,\frac{\sqrt{1-t^2+t^2+4t^2(1-t^2)}}{\sqrt{1-t^2}}\,dt \\ &= \int_0^1 t\sqrt{2-(1-2t^2)^2}\,dt = \frac{1}{4}\int_{-1}^1 \sqrt{2-u^2}du = \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2\theta\,d\theta \\ &= \frac{1}{4}\Big[\theta + \frac{\sin(2\theta)}{2}\Big]\Big|_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} = \frac{\pi}{8} + \frac{1}{4}\,.\end{aligned}$$

- **Measurements of the circulation - the curl operator**

We first talk about the rotation speed of a two-dimensional vector field. Imagine that on the plane there is a cylinder centered at (a,b) with radius r and millions of people is walking on this plane according to the velocity $\boldsymbol{u}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$; that is, when a people is at the location (x,y), then the people moves with the velocity $M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. When a person contacts the cylinder, the cylinder will rotate with the speed of that person. Since only the tangent speed contributes to the rotation of the cylinder, the "rotation speed" contributed by a person at location (x,y) is $(\boldsymbol{u}\cdot\mathbf{T})(x,y)$, where $\mathbf{T}$ is the unit tangent pointing in the counterclockwise direction (here a counterclockwise rotation is treated as rotation with positive speed and a clockwise rotation is treated as rotation with negative speed). A good way to measure the rotation speed is the angular velocity, so the angular velocity contributed by a person at location (x,y) is $\dfrac{(\boldsymbol{u}\cdot\mathbf{T})(x,y)}{r}$. Therefore, the average angular velocity of the cylinder is given by

$$\frac{1}{2\pi r}\oint_{C_r} \frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\, ds\,.$$

Since C_r can be parameterized by

$$\boldsymbol{r}(t) = (a + r\cos t)\mathbf{i} + (b + r\sin t)\mathbf{j}\,, \qquad t \in [0, 2\pi]\,,$$

by Theorem 15.17 we find that

$$\begin{aligned}
\frac{1}{2\pi r}\oint_{C_r} \frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\, ds &= \frac{1}{2\pi r^2}\int_0^{2\pi} (\boldsymbol{u}\cdot\mathbf{T})\big(\boldsymbol{r}(t)\big)\big\|\boldsymbol{r}'(t)\big\|\, dt = \frac{1}{2\pi r}\int_0^{2\pi} \boldsymbol{u}(\boldsymbol{r}(t))\cdot\mathbf{T}(\boldsymbol{r}(t))\, dt \\
&= \frac{1}{2\pi r}\int_0^{2\pi} \Big[M(a + r\cos t, b + r\sin t)\mathbf{i} + N(a + r\cos t, b + r\sin t)\mathbf{j}\Big]\cdot\big(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}\big)\, dt \\
&= \frac{1}{2\pi r}\int_0^{2\pi} \Big[N(a + r\cos t, b + r\sin t)\cos t - M(a + r\cos t, b + r\sin t)\sin t\Big]\, dt\,.
\end{aligned}$$

We then consider the limit of the average angular velocity as $r \to 0$. This limit, if it exists, is called the rotation speed of the velocity $\boldsymbol{u}$ at (a,b) which can be viewed as the angular velocity of an axis perpendicular to the plane and passing through (a,b). Define

$$f(r) = \int_0^{2\pi} \Big[N(a + r\cos t, b + r\sin t)\cos t - M(a + r\cos t, b + r\sin t)\sin t\Big]\, dt\,.$$

Then the rotation speed of the axis through (a,b) is $\lim\limits_{r\to 0}\dfrac{f(r)}{2\pi r}$. Note that if M, N is continuous, then

$$\lim_{r\to 0} f(r) = \int_0^{2\pi} \Big[N(a,b)\cos t - M(a,b)\sin t\Big]\, dt = 0\,;$$

thus if in addition M, N are differentiable, we are able to apply L'Hôpital's rule to find the limit: since the chain rule shows that

$$
\begin{aligned}
f'(0) &= \int_0^{2\pi} \frac{\partial}{\partial r}\Big|_{r=0} \Big[N(a + r\cos t, b + r\sin t)\cos t - M(a + r\cos t, b + r\sin t)\sin t\Big]\, dt \\
&= \int_0^{2\pi} \Big[\big(N_x(a, b)\cos t + N_y(a, b)\sin t\big)\cos t - \big(M_x(a, b)\cos t + M_y(a, b)\sin t\big)\sin t\Big]\, dt \\
&= \int_0^{2\pi} \Big[N_x(a, b)\cos^2 t + N_y(a, b)\sin t\cos t - M_x(a, b)\sin t\cos t - M_y(a, b)\sin^2 t\Big]\, dt \\
&= \pi\big[N_x(a, b) - M_y(a, b)\big]\,,
\end{aligned}
$$

we conclude that the rotation speed of the velocity $\boldsymbol{u}$ at (a, b) is

$$
\begin{aligned}
&\lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} \Big[N(a + r\cos t, b + r\sin t)\cos t - M(a + r\cos t, b + r\sin t)\sin t\Big]\, dt \\
&\qquad = \frac{1}{2}\Big[\frac{\partial N}{\partial x}(a, b) - \frac{\partial M}{\partial y}(a, b)\Big] = \frac{1}{2}(\mathrm{curl}\boldsymbol{u})(a, b)\,.
\end{aligned}
$$

We now consider the circulation or the speed of rotation of a three-dimension vector field $\boldsymbol{u}$ about an axis in the direction $\mathbf{N}$. Let P be a plane passing thorough a point a and having normal $\mathbf{N}$, and C_r be a circle on the plane P centered at a with radius r. Pick the orientation of the unit tangent vector $\mathbf{T}$ which is compatible with the unit normal $\mathbf{N}$ (see Figure 15.1 for reference).

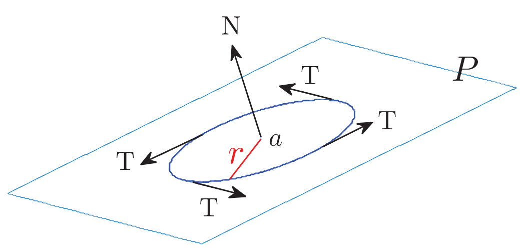

Figure 15.1: the circulation about an axis in direction $\mathbf{N}$

As illustrated above, the angular velocity of a vector field $\boldsymbol{u}$ along the circle C_r is measured by $\dfrac{\boldsymbol{u}\cdot\mathbf{T}}{r}$, it is quite reasonable to measure the circulation of u along C_r by averaging the angular velocity; that is, we consider the quantity

$$\frac{1}{2\pi r}\oint_{C_r} \frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\, ds \tag{15.2.4}$$

as a (constant multiple of) measurement of the speed of rotation. The limit of the quantity above, as $r \to 0$, is then a good measurement of the rotation speed of $\boldsymbol{u}$ at the point a about the axis in the direction $\mathbf{N}$.

We start from the case that $\mathbf{N} = \mathbf{e}_3$ so that P be parallel to the x_1x_2-plane. With u_1, u_2, u_3 denoting respectively the first, the second and the third components of $\boldsymbol{u}$, by the change of variable $ds = rd\theta$ and L'Hôpital's rule (to obtain the second "$=$") we find that

$$
\begin{aligned}
&\lim_{r\to 0} \frac{1}{2\pi r} \oint_{C_r} \frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\, ds \\
&\quad = \lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} \big[u_2\big(\boldsymbol{a} + (r\cos t, r\sin t, 0)\big)\cos t - u_1\big(\boldsymbol{a} + (r\cos t, r\sin t, 0)\big)\sin t\big]\, dt \\
&\quad = \frac{1}{2\pi}\frac{d}{dr}\Big|_{r=0} \int_0^{2\pi} \big[u_2\big(\boldsymbol{a} + (r\cos t, r\sin t, 0)\big)\cos t - u_1\big(\boldsymbol{a} + (r\cos t, r\sin t, 0)\big)\sin t\big]\, dt \\
&\quad = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r}\Big|_{r=0} \big[u_2\big(\boldsymbol{a} + (r\cos t, r\sin t, 0)\big)\cos t - u_1\big(\boldsymbol{a} + (r\cos t, r\sin t, 0)\big)\sin t\big]\, dt \\
&\quad = \frac{1}{2\pi} \int_0^{2\pi} \Big[\frac{\partial u_2}{\partial x_1}(\boldsymbol{a})\cos^2 t + \frac{\partial u_2}{\partial x_2}(\boldsymbol{a})\cos t\sin t - \frac{\partial u_1}{\partial x_1}(\boldsymbol{a})\cos t\sin t - \frac{\partial u_1}{\partial x_2}(\boldsymbol{a})\sin^2 t\Big]\, dt \\
&\quad = \frac{1}{2}\Big[\frac{\partial u_2}{\partial x_1}(\boldsymbol{a}) - \frac{\partial u_1}{\partial x_2}(\boldsymbol{a})\Big].
\end{aligned} \tag{15.2.5}
$$

Now suppose the general case that $\mathbf{N} \neq \mathbf{e}_3$. Let $\widehat{\mathbf{e}}_3 = \mathbf{N}$ and choose $\widehat{\mathbf{e}}_1$ and $\widehat{\mathbf{e}}_2$ so that $\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3\}$ is an orthonormal basis following the right-hand rule (that is, $\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2 = \widehat{\mathbf{e}}_3$). Then the vector field $\boldsymbol{u}$ has two representations

$$
\boldsymbol{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = v_1\widehat{\mathbf{e}}_1 + v_2\widehat{\mathbf{e}}_2 + v_3\widehat{\mathbf{e}}_3\,, \tag{15.2.6}
$$

here we use $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ instead of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Introduce a new Cartesian coordinate system $\boldsymbol{y} = (y_1, y_2, y_3)$ so that

$$
y_1\widehat{\mathbf{e}}_1 + y_2\widehat{\mathbf{e}}_2 + y_3\widehat{\mathbf{e}}_3 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3\,.
$$

In other words, $\boldsymbol{y}$ is the coordinate with coordinate axis parallel to the basis $\{\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3\}$; thus $\boldsymbol{y} = \mathrm{O}^{\mathrm{T}}\boldsymbol{x}$, where $\mathrm{O} = \big[\widehat{\mathbf{e}}_1 \vdots \widehat{\mathbf{e}}_2 \vdots \widehat{\mathbf{e}}_3\big]$. In this new Cartesian coordinate system, (15.2.5) implies that

$$
\lim_{r\to 0} \frac{1}{2\pi r} \oint_{C_r} \frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\, ds = \frac{1}{2}\Big[\frac{\partial v_2}{\partial y_1}(\boldsymbol{b}) - \frac{\partial v_1}{\partial y_2}(\boldsymbol{b})\Big]\,,
$$

where $\boldsymbol{b} = \mathrm{O}^{\mathrm{T}}\boldsymbol{a}$.

Now we transform the result above back to the original coordinate system (so that the limit is in terms of derivatives of u_j w.r.t. x_i). Note that (15.2.6) implies that $\boldsymbol{v} = \mathrm{O}^{\mathrm{T}}\boldsymbol{u}$ so that $v_j = \widehat{\mathbf{e}}_j \cdot \boldsymbol{u}$. Moreover, with e_{jk} denoting the k-th component (w.r.t. the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$) of $\widehat{\mathbf{e}}_j$; that is, $\widehat{\mathbf{e}}_j = e_{j1}\mathbf{e}_1 + e_{j2}\mathbf{e}_2 + e_{j3}\mathbf{e}_3$, the chain rule provides that

$$
\frac{\partial}{\partial y_1} = e_{11}\frac{\partial}{\partial x_1} + e_{12}\frac{\partial}{\partial x_2} + e_{13}\frac{\partial}{\partial x_3} \qquad \text{and} \qquad \frac{\partial}{\partial y_2} = e_{21}\frac{\partial}{\partial x_1} + e_{22}\frac{\partial}{\partial x_2} + e_{23}\frac{\partial}{\partial x_3}\,;
$$

thus

$$\lim_{r\to 0}\frac{1}{2\pi r}\oint_{C_r}\frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds=\frac{1}{2}\sum_{j=1}^{3}\Big[e_{1j}\frac{\partial(\boldsymbol{u}\cdot\widehat{\mathbf{e}}_2)}{\partial x_j}(\boldsymbol{a})-e_{2j}\frac{\partial(\boldsymbol{u}\cdot\widehat{\mathbf{e}}_1)}{\partial x_j}(\boldsymbol{a})\Big]$$
$$=\frac{1}{2}\sum_{j,k=1}^{3}(e_{1j}e_{2k}-e_{2j}e_{1k})\frac{\partial u_k}{\partial x_j}(\boldsymbol{a})=\frac{1}{2}\sum_{j,k,r,s=1}^{3}(\delta_{jr}\delta_{ks}-\delta_{js}\delta_{kr})e_{1r}e_{2s}\frac{\partial u_k}{\partial x_j}(\boldsymbol{a})\,,$$

where $\delta_{\cdot\cdot}$'s are the Kronecker deltas. Using (10.2.2), we further conclude that

$$\lim_{r\to 0}\frac{1}{2\pi r}\oint_{C_r}\frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds=\frac{1}{2}\sum_{i,j,k,r,s=1}^{3}\varepsilon_{ijk}\varepsilon_{irs}e_{1r}e_{2s}\frac{\partial u_k}{\partial x_j}(\boldsymbol{a})\,.$$

Since $\widehat{\mathbf{e}}_1\times\widehat{\mathbf{e}}_2=\widehat{\mathbf{e}}_3$, we have $e_{3i}=\sum\limits_{r,s=1}^{3}\varepsilon_{irs}e_{1r}e_{2s}$; thus the identity above shows that

$$\lim_{r\to 0}\frac{1}{2\pi r}\oint_{C_r}\frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds=\frac{1}{2}\sum_{i,j,k=1}^{3}\varepsilon_{ijk}e_{3i}\frac{\partial u_k}{\partial x_j}(\boldsymbol{a})=\frac{1}{2}\sum_{i=1}^{3}\Big(\sum_{j,k=1}^{3}\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}(\boldsymbol{a})\Big)e_{3i}=\frac{\mathrm{curl}\boldsymbol{u}\cdot\mathbf{N}}{2}\,.$$

Therefore, we conclude that

> the rotation speed of a three-dimensional vector field $\boldsymbol{u}$ at point $\boldsymbol{a}$ about the axis in the direction $\mathbf{N}$ is $\dfrac{\mathrm{curl}\boldsymbol{u}\cdot\mathbf{N}}{2}$.

15.2.2 Line integrals of vector fields

In the previous section we have seen the line integral

$$\frac{1}{2\pi r}\oint_{C}\frac{\boldsymbol{u}\cdot\mathbf{T}}{r}\,ds$$

which stands for the average angular velocity of the fluid velocity $\boldsymbol{u}$. In many applications, for a vector field $\boldsymbol{F}$ defined on a smooth curve C the line integral of the scalar function $\boldsymbol{F}\cdot\mathbf{T}$ along C, where $\mathbf{T}$ points to a given direction (there are two choices of directions if $\mathbf{T}$ is continuous on C), is considered. The vector field $\boldsymbol{F}$ here is often considered as a force field which acts on objects moving along C in the direction $\mathbf{T}$ so that the line integral

$$\int_C \boldsymbol{F}\cdot\mathbf{T}\,ds$$

is the work done by the force field $\boldsymbol{F}$. Every smooth curve admits two continuous tangent vector, and each tangent direction gives an ***orientation*** of a curve. This induces the following

Definition 15.22

An oriented curve is a curve on which a consistent tangent direction $\mathbf{T}$ is defined. In other words, an oriented curve is a (piecewise) smooth curve with a given parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ so that $\mathbf{T} = \dfrac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|}$ is defined.

Definition 15.23

Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth oriented curve C parameterized by $\boldsymbol{r}(t)$ for $t \in [a, b]$. The line integral of F along C is given by

$$\int_C \boldsymbol{F} \cdot \mathbf{T}\, ds\,.$$

Note that since $\mathbf{T} \circ \boldsymbol{r} = \dfrac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|}$, by Theorem 15.17 we have

$$\int_C \boldsymbol{F} \cdot \mathbf{T}\, ds = \int_a^b (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} \|\boldsymbol{r}'(t)\|\, dt = \int_a^b (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt\,.$$

Since $\boldsymbol{r}'(t)\, dt = d\boldsymbol{r}(t)$, sometimes we also use $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ to denote the line integral of $\boldsymbol{F}$ along the oriented curve C parameterized by $\boldsymbol{r}$.

Remark 15.24. Given an oriented curve C and $\boldsymbol{F} : C \to \mathbb{R}^3$, we sometimes use $\displaystyle\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r}$ to denote the line integral $\displaystyle\int_C \boldsymbol{F} \cdot (-\mathbf{T})\, ds$, where $-\mathbf{T}$ is the tangent direction opposite to the orientation of C.

Example 15.25. Find the work done by the force field

$$\boldsymbol{F}(x, y, z) = -\frac{1}{2}x\mathbf{i} - \frac{1}{2}y\mathbf{j} + \frac{1}{4}\mathbf{k}$$

on a particle as it moves along the helix parameterized by

$$\boldsymbol{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$$

from the point $(1, 0, 0)$ to the point $(-1, 0, 3\pi)$. Note that such a helix is parameterized by $\boldsymbol{r}(t)$ with $t \in [0, 3\pi]$. Therefore,

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{3\pi} \Big(-\frac{1}{2}\cos t\,\mathbf{i} - \frac{1}{2}\sin t\,\mathbf{j} + \frac{1}{4}\mathbf{k}\Big) \cdot \big(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}\big)\, dt \\ &= \int_0^{3\pi} \Big(\frac{1}{2}\sin t \cos t - \frac{1}{2}\sin t \cos t + \frac{1}{4}\Big)\, dt = \frac{3\pi}{4}\,.\end{aligned}$$

Example 15.26. Let $\boldsymbol{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ from $(0, 0)$ to $(1, 1)$ along

1. the straight line $y = x$,
2. the curve $y = x^2$, and
3. the piecewise smooth path consisting of the straight line segments from $(0, 0)$ to $(0, 1)$ and from $(0, 1)$ to $(1, 1)$.

For the straight line case, we parameterize the path by $\boldsymbol{r}(t) = (t, t)$ for $t \in [0, 1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t^2\mathbf{i} + 2t^2\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})dt = \int_0^1 3t^2 dt = 1\,.$$

For the case of parabola, we parameterize the path by $\boldsymbol{r}(t) = (t, t^2)$ for $t \in [0, 1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t^4\mathbf{i} + 2t^3\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})dt = \int_0^1 5t^4 dt = 1\,.$$

For the piecewise linear case, we let C_1 denote the line segment joining $(0, 0)$ and $(0, 1)$, and let C_2 denote the line segment joining $(0, 1)$ and $(1, 1)$. Note that we can parameterize C_1 and C_2 by

$$\boldsymbol{r}_1(t) = t\mathbf{j} \quad t \in [0, 1] \quad \text{and} \quad \boldsymbol{r}_2(t) = t\mathbf{i} + \mathbf{j} \quad t \in [0, 1]\,,$$

respectively. Therefore,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 t^2\mathbf{i} \cdot \mathbf{j}\, dt + \int_0^1 (\mathbf{i} + 2t\mathbf{j}) \cdot \mathbf{i}\, dt = 1\,.$$

We note that in this example the line integrals of $\boldsymbol{F}$ along three different paths joining $(0, 0)$ and $(1, 1)$ are identical.

Example 15.27. Let $\boldsymbol{F}(x, y) = y\mathbf{i} - x\mathbf{j}$. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ from $(1, 0)$ to $(0, -1)$ along

1. the straight line segment joining these points, and
2. three-quarters of the circle of unit radius centered at the origin and traversed counter-clockwise.

For the first case, we parameterize the path by $\boldsymbol{r}(t) = (1-t, -t)$ for $t \in [0,1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 \big[-t\mathbf{i} + (t-1)\mathbf{j}\big] \cdot (-\mathbf{i} - \mathbf{j})\, dt = \int_0^1 1\, dt = 1\,.$$

For the second case, we parameterize the path by $\boldsymbol{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $t \in \big[0, \frac{3\pi}{2}\big]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{\frac{3\pi}{2}} (\sin t\,\mathbf{i} - \cos t\,\mathbf{j}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\, dt = \int_0^{\frac{3\pi}{2}} (-1)\, dt = -\frac{3\pi}{2}\,.$$

We note that in this example the line integrals of $\boldsymbol{F}$ along different paths joining $(1,0)$ and $(0,-1)$ can be different.

15.3 Conservative Vector Fields and Independence of Path

In the previous section, we define the line integral of a force along a curve in a given orientation. In Example 15.26, we see that the line integrals along three different paths connecting two given points are the same, while in Example 15.27 the line integrals along two different paths (connecting two given points) are different. In this section, we are interested in the rule of judging whether the line integral is path independent or not.

Theorem 15.28: Fundamental Theorem of Line Integrals

Let C be a piecewise smooth curve in an open region $\mathcal{D}$ parameterized by $\boldsymbol{r} : [a,b] \to \mathbb{R}^3$. If $\boldsymbol{F}$ is a conservative vector field in $\mathcal{D}$ and ϕ is a continuous differentiable potential for $\boldsymbol{F}$, then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \phi\big(\boldsymbol{r}(b)\big) - \phi\big(\boldsymbol{r}(a)\big)\,.$$

Proof. Suppose that $\{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a,b]$ such that $\boldsymbol{r}$ is continuously differentiable on $[t_{i-1}, t_i]$ for each $1 \leqslant i \leqslant n$. Then the chain rule implies that

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\nabla\phi)(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{d}{dt}(\phi \circ \boldsymbol{r})(t)\, dt = \sum_{i=1}^n (\phi \circ \boldsymbol{r})(t)\Big|_{t=t_{i-1}}^{t=t_i} = \phi(\boldsymbol{r}(b)) - \phi(\boldsymbol{r}(a))\end{aligned}$$

which concludes the theorem. $\square$

Remark 15.29. If $\boldsymbol{F}$ is a conservative vector field and ϕ is a potential for $\boldsymbol{F}$, then the Fundamental Theorem of Line Integrals indeed shows that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \phi(\text{the end-point of } C) - \phi(\text{the starting point of } C)\,.$$

Moreover, in the proof of Theorem 15.28 the parametrization $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ does not has to be injective (so the image of $\boldsymbol{r}$ might have a lot of overlapping). It is only required that the parametrization is piecewise smooth for the proof to work.

Example 15.30 (Revisit of Example 15.26)**.** Suppose that $\boldsymbol{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ and C is an oriented piecewise smooth curve joining $(0, 0)$ and $(1, 1)$, where $(0, 0)$ is the starting point and $(1, 1)$ is the end-point of the oriented curve. Since $\boldsymbol{F}$ is conservative with potential $\phi(x, y) = xy^2$, by the Fundamental Theorem of Line Integrals we conclude that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \phi(1, 1) - \phi(0, 0) = 1\,.$$

Example 15.31. Evaluate $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is a piecewise smooth curve from $(1, 1, 0)$ to $(0, 2, 3)$ and

$$\boldsymbol{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$$

as given in Example 15.4. Recall that this $\boldsymbol{F}$ has a potential $\phi(x, y, z) = x^2y + yz^2$ as explained in Example 15.4. Therefore, the Fundamental Theorem of Line Integrals implies that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \phi(0, 2, 3) - \phi(1, 1, 0) = 17.$$

Theorem 15.32

Let $\mathcal{D}$ be an open, connected domain in $\mathbb{R}^3$, and let $\mathbf{F}$ be a smooth vector field defined on $\mathcal{D}$. Then the following three statements are equivalent:

(1) $\mathbf{F}$ is conservative in $\mathcal{D}$.

(2) $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth, closed curve C in $\mathcal{D}$.

(3) Given any two point $\mathbf{p}_0, \mathbf{p}_1 \in \mathcal{D}$, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all piecewise smooth curves in $\mathcal{D}$ starting at $\mathbf{p}_0$ and ending at $\mathbf{p}_1$.

In the statement of the theorem, the (path) connectedness of a region means that any two points in the region can be joined by a piecewise smooth curve lying entirely within the region.

Proof of Theorem 15.32. (1) $\Rightarrow$ (2): This is a direct consequence of the Fundamental Theorem of Line Integrals.

(2) $\Rightarrow$ (3): Let C_1 and C_2 be two piecewise smooth curves in $\mathcal{D}$ starting at $\mathbf{p}_0$ and ending at $\mathbf{p}_1$ parameterized by $\boldsymbol{r}_1 : [a,b] \to \mathbb{R}^3$ and $\boldsymbol{r}_2 : [c,d] \to \mathbb{R}^3$, respectively. Define $\boldsymbol{r} : [a, b+d-c] \to \mathbb{R}^3$ by

$$\boldsymbol{r}(t) = \begin{cases} \boldsymbol{r}_1(t) & \text{if } t \in [a,b]\,, \\ \boldsymbol{r}_2(b+d-t) & \text{if } t \in [b, b+d-c]\,. \end{cases}$$

Then $C = \boldsymbol{r}([a, b+d-c])$ is a piecewise smooth closed curve; thus

$$\begin{aligned} 0 = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_a^b (\boldsymbol{F} \circ \boldsymbol{r}_1)(t) \cdot \boldsymbol{r}_1'(t)\, dt - \int_b^{b+d-c} (\boldsymbol{F} \circ \boldsymbol{r}_2)(b+d-t) \boldsymbol{r}_2'(b+d-t)\, dt \\ &= \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} - \int_c^d (\boldsymbol{F} \circ \boldsymbol{r}_2)(t) \boldsymbol{r}_2'(t) dt = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} - \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}\,. \end{aligned}$$

(3) $\Rightarrow$ (1): Let $\mathbf{p}_0 \in \mathcal{D}$. For $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathcal{D}$, define $\phi(\boldsymbol{x}) = \displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is any piecewise smooth curve starting at $\mathbf{p}_0$ and ending at $\boldsymbol{x}$. Note that by assumption, $\phi : \mathcal{D} \to \mathbb{R}$ is well-defined.

Choose $\delta > 0$ such that $B(\boldsymbol{x}, \delta) \subseteq \mathcal{D}$, where $B(\boldsymbol{x}, \delta)$ is the ball centered at $\boldsymbol{x}$ with radius δ. Let C be a piecewise smooth curve joining $\mathbf{p}_0$, and L be the line segment joining $\boldsymbol{x}$ and $\boldsymbol{x} + h\mathbf{e}_j$, where $0 < h < \delta$ and $\mathbf{e}_j$ is the unit vector whose j-th component is 1. Then with the parametrization of L: $\boldsymbol{r}(t) = \boldsymbol{x} + t\mathbf{e}_j$ for $t \in [0,h]$ (or $[h,0]$ if $h < 0$), we have

$$\frac{\phi(\boldsymbol{x} + h\mathbf{e}_j) - \phi(\boldsymbol{x})}{h} = \frac{1}{h} \int_L \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{1}{h} \int_0^h \boldsymbol{F}(\boldsymbol{x} + t\mathbf{e}_j) \cdot \mathbf{e}_j\, dt\,;$$

thus passing to the limit as $h \to 0$, we find that

$$\frac{\partial \phi}{\partial x_j}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}) \cdot \mathbf{e}_j\,.$$

As a consequence, $\boldsymbol{F} = \nabla \phi$. Moreover, since $\boldsymbol{F}$ is smooth, Theorem 13.35 implies that ϕ is differentiable on $\mathcal{D}$. Therefore, $\boldsymbol{F}$ is conservative. $\square$

Theorem 15.33: Law of Conservation of Energy

In a conservative force field, the sum of the potential and kinetic energies of an object remains constant from point to point. Here

1. the potential energy is the negative of the potential function of the conservative vector field;
2. the kinetic energy is $\frac{1}{2}m\|\boldsymbol{v}\|^2$, where m, $\boldsymbol{v}$ are the mass and the velocity of the object, respectively.

Proof. Let $\boldsymbol{F}$ be the conservative force field, p is the potential energy so that $-\nabla p = \boldsymbol{F}$, and k is the kinetic energy defined by $k = \frac{1}{2}m|\boldsymbol{v}|^2 = \frac{1}{2}m|\boldsymbol{r}'|^2$. Suppose that the curve C is parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ so that $\boldsymbol{r}(a) = A$ and $\boldsymbol{r}(b) = B$. Then

1. the work done by $\boldsymbol{F}$ along a smooth curve C from point A to point B is

$$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = (-p)(B) - (-p)(A) = p(A) - p(B)\,.$$

2. by Newton's second law of motion, $\boldsymbol{F} = m\boldsymbol{r}''$; thus the work done by $\boldsymbol{F}$ is

$$\begin{aligned} W &= \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b m\boldsymbol{r}''(t) \cdot \boldsymbol{r}'(t)\,dt = \int_a^b m\frac{1}{2}\frac{d}{dt}\big\|\boldsymbol{r}'(t)\big\|^2\,dt = \frac{m}{2}\int_a^b \frac{d}{dt}\big\|\boldsymbol{r}'(t)\big\|^2\,dt \\ &= \frac{m}{2}\big\|\boldsymbol{r}'(t)\big\|^2\Big|_{t=a}^{t=b} = \frac{m}{2}\big\|\boldsymbol{v}(t)\big\|^2\Big|_{t=a}^{t=b} = k(B) - k(A)\,. \end{aligned}$$

Therefore, $p(A) - p(B) = k(B) - k(A)$ or equivalently, $p(A) + k(A) = p(B) + k(B)$ which shows that $p + k$, the sum of the potential and kinetic energies, are constant from point to point. □

15.4 Green's Theorem

Even though Theorem 15.32 provides several equivalence for conservative vector fields, none of them is practically useful for determining whether a vector field is conservative since in reality it is very hard to compute all possible line integrals. Some other criteria (sufficient conditions) for determining conservative vector fields have to be developed.

We first look at the case of two-dimensional vector fields. Let $\mathcal{D} \subseteq \mathbb{R}^2$, and $\boldsymbol{F} = (M, N) : \mathcal{D} \to \mathbb{R}^2$. If $\boldsymbol{F}$ is conservative, then $M = \phi_x$ and $N = \phi_y$ for some scalar function $\phi : \mathcal{D} \to \mathbb{R}$;

thus if ϕ is of class $\mathscr{C}^2$, we must have $M_y = N_x$. In other words, if $\boldsymbol{F} : \mathcal{D} \to \mathbb{R}^2$ is a smooth vector field, then it is necessary that $M_y = N_x$. The converse statement is not true in general, and we have the following counter-example.

Example 15.34. Let $\mathcal{D} \subseteq \mathbb{R}^2$ be the annular region $\mathcal{D} = \big\{(x, y) \,\big|\, 1 < x^2 + y^2 < 4\big\}$, and consider the vector field $\boldsymbol{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$. Then

$$\frac{\partial}{\partial y}\frac{y}{x^2 + y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial}{\partial x}\frac{-x}{x^2 + y^2};$$

however, if $\boldsymbol{F} = \nabla\phi$ for some differentiable scalar function $\phi : \mathcal{D} \to \mathbb{R}$, we must have

$$\phi_x(x, y) = \frac{y}{x^2 + y^2}$$

which further implies that

$$\phi(x, y) = \arctan\frac{x}{y} + f(y)\,.$$

Using that $\phi_y(x, y) = -\dfrac{x}{x^2 + y^2}$, we conclude that f is a constant function; thus

$$\phi(x, y) = \arctan\frac{x}{y} + C\,.$$

Since ϕ is not differentiable on the positive x-axis, $\boldsymbol{F} \neq \nabla\phi$.

We can also consider the line integral $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is the oriented circle parameterized by

$$\boldsymbol{r}(t) = \sqrt{2}\cos t\,\mathbf{i} + \sqrt{2}\sin t\,\mathbf{j} \qquad t \in [0, 2\pi]\,.$$

Using the formula to compute the line integral, we find that

$$\begin{aligned}\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{2\pi} \Big(\frac{\sqrt{2}\sin t}{2}\mathbf{i} - \frac{\sqrt{2}\cos t}{2}\mathbf{j}\Big) \cdot \big(-\sqrt{2}\sin t\,\mathbf{i} + \sqrt{2}\cos t\,\mathbf{j}\big)\,dt \\ &= \int_0^{2\pi} (-1)\,dt = -2\pi\,;\end{aligned}$$

thus Theorem 15.32 implies that $\boldsymbol{F}$ cannot be conservative.

Let $R \subseteq \mathbb{R}^2$ be a region enclosed by a simply closed curve C and $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$ be a vector fields on (an open set containing) R, where C is ***oriented counterclockwise*** so that

C is traversed once so that the region R always lies to the left.

The line integral of $\boldsymbol{F}$ along an oriented curve C sometimes is written as

$$\int_C M dx + N dy$$

since symbolically we have $d\boldsymbol{r} = dx\mathbf{i} + dy\mathbf{j}$ so that

$$\boldsymbol{F} \cdot d\boldsymbol{r} = (M\mathbf{i} + N\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = M dx + N dy \,.$$

The right-hand side of the identity above is called a ***differential form***.

Suppose that

1. Every vertical line intersects C at at most two points so that

$$R = \left\{(x, y) \,\middle|\, x \in [a, b], f_1(x) \leqslant y \leqslant f_2(x)\right\}.$$

2. Every horizontal line intersects C at at most two points so that

$$R = \left\{(x, y) \,\middle|\, y \in [c, d], g_1(y) \leqslant x \leqslant g_2(y)\right\}.$$

In other words, the curve C is the union of

1. C_1 and C_2 parameterized respectively by

$$\boldsymbol{r}_1(t) = t\mathbf{i} + f_1(t)\mathbf{j}\,, \quad t \in [a, b]\,, \qquad \text{and} \qquad \boldsymbol{r}_2(t) = -t\mathbf{i} + f_2(-t)\mathbf{j}\,, \quad t \in [-b, -a]\,.$$

2. C_3 and C_3 parameterized respectively by

$$\boldsymbol{r}_3(t) = g_2(t)\mathbf{i} + t\mathbf{j}\,, \quad t \in [c, d] \qquad \text{and} \qquad \boldsymbol{r}_4(t) = g_1(-t)\mathbf{i} - t\mathbf{j}\,, \quad t \in [-d, -c]\,.$$

We note that C_1, C_2, C_3 and C_4 are oriented curves. Then by the definition of the line integrals,

$$\oint_C (\boldsymbol{F}_1 + \boldsymbol{F}_2) \cdot d\boldsymbol{r} = \oint_C \boldsymbol{F}_1 \cdot d\boldsymbol{r} + \oint_C \boldsymbol{F}_2 \cdot d\boldsymbol{r}\,;$$

thus

$$\begin{aligned}\oint_C M dx + N dy = \oint_C (M\mathbf{i} + N\mathbf{j}) \cdot d\boldsymbol{r} = \oint_C M\mathbf{i} \cdot d\boldsymbol{r} + \oint_C N\mathbf{j} \cdot d\boldsymbol{r} \\ = \int_{C_1} M\mathbf{i} \cdot d\boldsymbol{r} + \int_{C_2} M\mathbf{i} \cdot d\boldsymbol{r} + \int_{C_3} N\mathbf{j} \cdot d\boldsymbol{r} + \int_{C_4} N\mathbf{j} \cdot d\boldsymbol{r}\,.\end{aligned}$$

Using the formula for computing the line integrals, we find that

1. $\displaystyle\int_{C_1} M\,\mathbf{i}\cdot d\boldsymbol{r} = \int_a^b M(t, f_1(t))\,dt$, and $\displaystyle\int_{C_3} N\,\mathbf{j}\cdot d\boldsymbol{r} = \int_c^d N(g_2(t), t)\,dt$.

2. $\displaystyle\int_{C_2} M\,\mathbf{i}\cdot d\boldsymbol{r} = -\int_{-b}^{-a} M(-t, f_2(-t))\,dt$, and $\displaystyle\int_{C_4} N\,\mathbf{j}\cdot d\boldsymbol{r} = -\int_{-d}^{-c} M(g_1(-t), -t)\,dt$; thus the substitution of variable $t \mapsto -t$ shows that

$$\int_{C_2} M\,\mathbf{i}\cdot d\boldsymbol{r} = -\int_a^b M(t, f_2(t))\,dt \quad \text{and} \quad \int_{C_4} N\,\mathbf{j}\cdot d\boldsymbol{r} = -\int_c^d M(g_1(t), t)\,dt\,.$$

Therefore, if M_y and N_x are continuous on R,

$$\begin{aligned}
\oint_C M dx + N dy &= \int_a^b \big[M(t, f_1(t)) - M(t, f_2(t))\big]\,dt + \int_c^d \big[N(g_2(t), t) - N(g_1(t), t))\big]\,dt \\
&= \int_a^b \big[M(x, f_1(x)) - M(x, f_2(x))\big]\,dx + \int_c^d \big[N(g_2(y), y) - N(g_1(y), y))\big]\,dy \\
&= -\int_a^b \Big(\int_{f_1(x)}^{f_2(x)} M_y(x, y)\,dy\Big)dx + \int_c^d \Big(\int_{g_1(y)}^{g_2(y)} N_x(x, y)\,dx\Big)dy
\end{aligned}$$

and the Fubini Theorem further implies that

$$\oint_C M dx + N dy = -\iint_R M_y(x, y)\,dA + \iint_R N_x(x, y)\,dA = \iint_R \big(N_x - M_y\big)(x, y)\,dA\,. \quad (15.4.1)$$

We note that (15.4.1) in particular implies (2) of Theorem 15.32 when C is a simply close plane curve of the form given above.

Identity (15.4.1) is in fact true as long as C is a closed plane curve oriented counterclockwise, and we have the following

Theorem 15.35: Green's Theorem

Let R be a plane region enclosed by a closed curve C oriented counterclockwise; that is, C is traversed once so that the region R always lies to the left. If M and N have continuous first partial derivatives in an open region containing R, then

$$\oint_C M dx + N dy = \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)(x, y)\,dA\,. \quad (15.4.1)$$

Remark 15.36. If $\boldsymbol{F}$ is a two-dimensional vector field given by $\boldsymbol{F} = M\,\mathbf{i} + N\,\mathbf{j}$, then under the assumption of Green's Theorem,

$$\oint_C \boldsymbol{F}\cdot\mathbf{T}\,ds = \iint_R (\mathrm{curl}\boldsymbol{F})(x, y)\,dA\,.$$

This is sometimes called **Green's Theorem in Tangential Form**. Moreover, by treating $\boldsymbol{F}$ as a three-dimensional vector field, then under the assumption of Green's Theorem,

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_R (\text{curl}\boldsymbol{F} \cdot \mathbf{k})(x, y)\, dA\,.$$

Remark 15.37. Let R be a region enclosed by a smooth simply closed curve C with **outward-pointing** unit normal $\mathbf{N}$ on C, and $\boldsymbol{F}$ be a smooth vector field defined on an open region containing R. We are interested in $\displaystyle\int_C \boldsymbol{F} \cdot \mathbf{N} ds$, the line integral of $\boldsymbol{F} \cdot \mathbf{N}$ along C.

Suppose that $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$, and C is parameterized by $\boldsymbol{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $t \in [a, b]$, so that C is oriented counterclockwise. Define $\boldsymbol{G} = -N\mathbf{i} + M\mathbf{j}$. Then Green's Theorem implies that

$$\oint_C - N dx - M dy = \oint_C \boldsymbol{G} \cdot d\boldsymbol{r} = \iint_R \text{curl}\boldsymbol{G}\, dA = \iint_R \big(M_x + N_y\big)\, dA = \iint_R \text{div}\boldsymbol{F}\, dA\,.$$

On the other hand, if $\boldsymbol{r}$ is a counterclockwise parametrization of C, then

$$\mathbf{N}(\boldsymbol{r}(t)) = \frac{y'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{i} + \frac{x'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{j} \qquad \forall\, t \in [a, b]\,;$$

thus

$$\begin{aligned}
\oint_C \boldsymbol{F} \cdot \mathbf{N}\, ds &= \int_a^b (\boldsymbol{F} \cdot \mathbf{N})(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\, dt = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \mathbf{N}(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\, dt \\
&= \int_a^b \big[M(x(t), y(t))\mathbf{i} + N(x(t), y(t))\mathbf{j}\big] \cdot \Big[\frac{y'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{i} + \frac{x'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{j}\Big]\|\boldsymbol{r}'(t)\|\, dt \\
&= \int_a^b \big[M(x(t), y(t))y'(t) - N(x(t), y(t))x'(t)\big]\, dt \\
&= \oint_C - N\, dx - M\, dy = \oint_C \boldsymbol{G} \cdot d\boldsymbol{r} = \iint_R \text{div}\boldsymbol{F}\, dA\,.
\end{aligned}$$

Therefore,

$$\int_C \boldsymbol{F} \cdot \mathbf{N}\, ds = \iint_R \text{div}\boldsymbol{F}\, dA\,.$$

This is sometimes called **Green's Theorem in Normal Form or Divergence Form**.

Example 15.38. Use Green' s Theorem to evaluate the line integral $\displaystyle\oint_C y^3 dx + (x^3 + 3xy^2) dy$, where C is the path from $(0, 0)$ to $(1, 1)$ along the graph of $y = x^3$ and from $(1, 1)$ to $(0, 0)$ along the graph of $y = x$.

Let $R = \{(x,y)\,\big|\,0 \leqslant x \leqslant 1, x^3 \leqslant y \leqslant x\}$. Then Grenn's Theorem implies that

$$\oint_C y^3 dx + (x^3 + 3xy^2)dy = \iint_R \Big[\frac{\partial}{\partial x}(x^3 + 3xy^2) - \frac{\partial}{\partial y} y^3\Big]\, dA = \iint_R 3x^2\, dA$$
$$= \int_0^1 \Big(\int_{x^3}^x 3x^2\, dy\Big)\, dx = \int_0^1 3x^2(x - x^3)\, dx = \Big(\frac{3}{4}x^4 - \frac{1}{2}x^6\Big)\Big|_{x=0}^{x=1} = \frac{1}{4}.$$

Example 15.39. Let $\boldsymbol{F}$ and $\mathcal{D}$ be given in Example 15.34, and $C \subseteq \mathcal{D}$ be a simple closed curve oriented counterclockwise so that the origin is inside the region enclosed by C. Find $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Choose $r > 1$ so that the circle centered at the origin with radius r lies in the region enclosed by C. Let C_r denote this circle with clockwise orientation, and pick a line segment B connecting C and C_r (with starting point on C and end-point on C_r. Define Γ as the oriented curve $B \cup C_r \cup (-B) \cup C$, where $-B$ denotes oriented curve B with opposite orientation, and let R be the region enclosed by Γ. Then $R \subseteq \mathcal{D}$ and R is the region lies to the left of Γ. Therefore, Green's Theorem implies that

$$\int_\Gamma \int_{C_r} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_R \operatorname{curl}\boldsymbol{F}\, dA = 0\,.$$

On the other hand,

$$\int_\Gamma \boldsymbol{F} \cdot d\boldsymbol{r} = \int_B \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_r} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{-B} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_C \boldsymbol{F} \cdot d\boldsymbol{r}\,;$$

thus by the fact that $\displaystyle\int_{-B} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_B \boldsymbol{F} \cdot d\boldsymbol{r}$, we conclude that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_r} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_\Gamma \int_{C_r} \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

or equivalently,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{C_r} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{-C_r} \boldsymbol{F} \cdot d\boldsymbol{r}\,.$$

In other words, the line integral of $\boldsymbol{F}$ along C is the same as the line integral of $\boldsymbol{F}$ along the circle C_r with counterclockwise orientation. Since $-C_r$ can be parameterized by

$$\boldsymbol{r}(t) = r\cos t\,\mathbf{i} + r\sin t\,\mathbf{j} \qquad t \in [0, 2\pi]\,,$$

we find that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{2\pi} \Big(\frac{r\sin t}{r^2}\mathbf{i} - \frac{r\cos t}{r^2}\mathbf{j}\Big) \cdot \big(-r\sin t\,\mathbf{i} + r\cos t\,\mathbf{j}\big)\, dt = \int_0^{2\pi} (-1)\, dt = -2\pi\,.$$

Definition 15.40

A connected region $\mathcal{D} \subseteq \mathbb{R}^2$ is said to be ***simply connected*** if every simple closed curve can be continuously shrunk to a point in $\mathcal{D}$ without any part ever passing out of $\mathcal{D}$. In other words, $\mathcal{D}$ is simply connected if every simple closed curve encloses a region which is a subset of $\mathcal{D}$.

Theorem 15.41

Let $\mathcal{D} \subseteq \mathbb{R}^2$ be simply connected, and M, N be functions defined on $\mathcal{D}$ with continuous first partial derivatives. If $M_y = N_x$ in $\mathcal{D}$, then $\boldsymbol{F} = M\,\mathbf{i} + N\,\mathbf{j}$ is conservative in $\mathcal{D}$.