Exercise Problem Sets 14

May 31. 2024

Problem 1. Evaluate the double integral $\iint\limits_R \arctan \frac{y}{x}\, dA$ using the polar coordinate, where

$$R = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 1 \leqslant x^2 + y^2 \leqslant 4, 0 \leqslant y \leqslant x\right\}.$$

Problem 2. Evaluate the triple integral $\iiint\limits_D x \exp(x^2 + y^2 + z^2)\, dV$, where D is the portion of the unit ball $x^2 + y^2 + z^2 \leqslant 1$ that lies in the first octant.

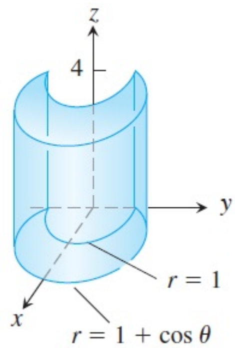
Problem 3. Evaluate the triple integral $\iiint\limits_D \sqrt{x^2 + y^2 + z^2}\, dV$, where D is the region lying above the cone $z = \sqrt{x^2 + y^2}$ and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

Problem 4. Use the cylinder coordinate to find the volume of the ball $x^2 + y^2 + z^2 = a^2$.

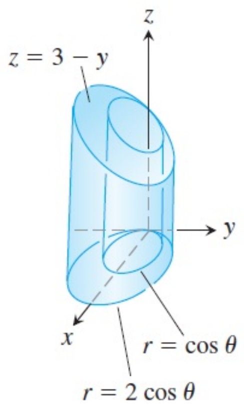
Problem 5. Use the spherical coordinate to find the volume of the cylindricality $x^2 + y^2 = r^2$, where $0 \leqslant z \leqslant h$.

Problem 6. Compute the volume of D given below using triple integrals in cylindrical coordinates.

(1) D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.

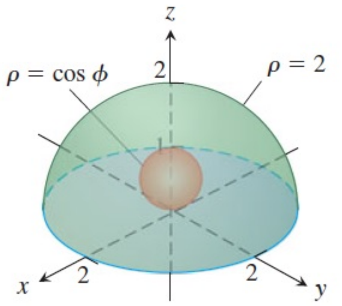


(2) D is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$.

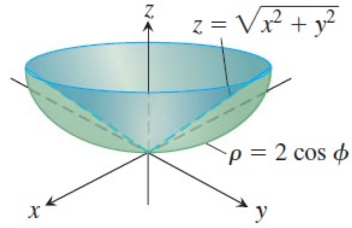


Problem 7. Compute the volume of D given below using triple integrals in spherical coordinates.

(1) D is the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, $z \geqslant 0$.



(2) D is the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$.



Problem 8. Convert the integral

$$\int_{-1}^{1} \Big[\int_{0}^{\sqrt{1-y^2}} \Big(\int_0^x (x^2 + y^2)\, dz \Big) dx \Big] dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result.

Problem 9. Find the integrals given below with specific change of variables.

(1) Find $\int_0^2 \Big(\int_{\frac{y}{2}}^{\frac{y+4}{2}} y^3(2x-y)e^{(2x-y)^2}\, dx \Big) dy$ using change of variables $x = u + \frac{1}{2}v$, $y = v$.

(2) Find $\iint\limits_{[0,1]\times[0,1]} \frac{1}{(1+xy)\ln(xy)}\, dA$ by making the change of variables $u = xy$ and $v = y$.

(3) Find $\int_1^2 \Big(\int_{\frac{1}{y}}^{y} (x^2+y^2)\, dx \Big) dy + \int_2^4 \Big(\int_{\frac{y}{4}}^{\frac{4}{y}} (x^2+y^2)\, dx \Big) dy$ using change of variables $x = \frac{u}{v}$, $y = uv$.

(4) Find $\int_0^1 \Big(\int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy \Big) dx$ using change of variables $x = u^2 - v^2$, $y = 2uv$.

(5) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Find $\iint\limits_R \Big(\sqrt{\frac{y}{x}} + \sqrt{xy} \Big)\, dA$ using the change of variables $x = \frac{u}{v}$, $y = uv$.

(6) Let D be the solid region in xyz-space defined by

$$D = \big\{ (x, y, z) \,\big|\, 1 \leqslant x \leqslant 2, 0 \leqslant xy \leqslant 2, 0 \leqslant z \leqslant 1 \big\}.$$

Find $\iiint\limits_D (x^2y + 3xyz)\, dV$ using change of variables $u = x$, $v = xy$, $w = 3z$.

Problem 10. Evaluate the double integral $\iint_R (x+y)e^{x^2-y^2}\,dA$, where R is rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

Problem 11. Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Show that

$$\iint_R f(x+y)\,dA = \int_0^1 uf(u)\,du\,.$$

Problem 12. Let A be the area of the region in the first quadrant bounded by the line $y=\frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2+y^2=1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y=mx$, the y-axis, and the ellipse $\frac{1}{9}x^2+y^2=1$.
Hint: Try to make change of variables so that the computation of the area of the region in the first quadrant bounded by the line $y=mx$, the y-axis, and the ellipse $\frac{1}{9}x^2+y^2=1$ looks the same as the former one.

Problem 13. Suppose that $\boldsymbol{c}\in\mathbb{R}^3$ is a unit vector; that is, $\|\boldsymbol{c}\|=1$. Show that

$$\iiint_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\boldsymbol{c}\cdot\boldsymbol{x})\,dV = 2\pi r-\pi\sin(2r)\,,$$

where $B(\mathbf{0},r)$ is the ball centered at the origin with radius r, by completing the following steps.

(1) Let O be an orthogonal 3×3 matrix (that is, $\mathrm{O}^{\mathrm{T}}\mathrm{O}=\mathrm{O}\mathrm{O}^{\mathrm{T}}=\mathrm{I}_{3\times 3}$). Show that the change of variable $\boldsymbol{x}=\mathrm{O}\boldsymbol{y}$ implies that

$$\iiint_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\boldsymbol{c}\cdot\boldsymbol{x})\,dV = \iiint_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\mathrm{O}^{\mathrm{T}}\boldsymbol{c}\cdot\boldsymbol{x})\,dV\,.$$

Hint: What is the Jacobian $\dfrac{\partial(x_1,x_2,x_3)}{\partial(y_1,y_2,y_3)}$?

(2) Use the fact that there exists an orthogonal 3×3 matrix O such that $\mathrm{O}^{\mathrm{T}}\boldsymbol{c}=(0,0,1)^{\mathrm{T}}$ to conclude that

$$\iiint_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\boldsymbol{c}\cdot\boldsymbol{x})\,dV = \iiint_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos x_3\,dV\,. \qquad (0.1)$$

(3) Use the spherical coordinates to computer the triple integral on the right-hand side of (0.1) (in the order $d\theta\,d\phi\,dr$) and obtain the desired result.