Exercise Problem Sets 12

May 19. 2024

Problem 1. Find the surface area for the portion of the surface $z = xy$ that is inside the cylinder $x^2 + y^2 = 1$.

Problem 2. Let Σ be a parametric surface parameterized by

$$\boldsymbol{r}(u,v) = X(u,v)\boldsymbol{i} + Y(u,v)\boldsymbol{j} + Z(u,v)\boldsymbol{k}\,, \quad (u,v) \in R\,.$$

Define $E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u$, $F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v$ and $G = \boldsymbol{r}_v \cdot \boldsymbol{r}_v$. Show that

$$\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|^2 = EG - F^2\,.$$

Hint: You can try to make use of ε_{ijk}, the permutation symbol.

Proof. Write $\boldsymbol{r} = \sum\limits_{i=1}^{3} R_i \boldsymbol{e}_i$, where $R_1 = X$, $R_2 = Y$, $R_3 = Z$ and $\boldsymbol{e}_1 = \boldsymbol{i}$, $\boldsymbol{e}_2 = \boldsymbol{j}$, $\boldsymbol{e}_3 = \boldsymbol{k}$,. Then

$$\boldsymbol{r}_u \equiv \sum_{j=1}^{3} \frac{\partial R_j}{\partial u} \boldsymbol{e}_j \qquad \text{and} \qquad \boldsymbol{r}_v \equiv \sum_{k=1}^{3} \frac{\partial R_k}{\partial v} \boldsymbol{e}_k\,.$$

By the fact that

$$(\boldsymbol{u} \times \boldsymbol{v}) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_j v_k \boldsymbol{e}_i \quad \text{if } \boldsymbol{u} = \sum_{j=1}^{3} u_j \boldsymbol{e}_j,\ \boldsymbol{v} = \sum_{k=1}^{3} v_k \boldsymbol{e}_k, ,$$

we find that

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} \boldsymbol{e}_i$$

so that

$$\begin{aligned}
\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|^2 &= \Big(\sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} \boldsymbol{e}_i \Big) \cdot \Big(\sum_{\ell,r,s=1}^{3} \varepsilon_{\ell rs} \frac{\partial R_r}{\partial u} \frac{\partial R_s}{\partial v} \boldsymbol{e}_\ell \Big) \\
&= \sum_{i,j,k,\ell,r,s=1}^{3} \varepsilon_{ijk}\varepsilon_{\ell rs} \frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} \frac{\partial R_r}{\partial u} \frac{\partial R_s}{\partial v} (\boldsymbol{e}_i \cdot \boldsymbol{e}_\ell) \\
&= \sum_{i,j,k,r,s=1}^{3} \varepsilon_{ijk}\varepsilon_{irs} \frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} \frac{\partial R_r}{\partial u} \frac{\partial R_s}{\partial v}\,.
\end{aligned}$$

Using the identity $\sum\limits_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$ (here $\delta_{..}$ denotes the Kronecker delta) we conclude that

$$\begin{aligned}
\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|^2 &= \sum_{j,k,r,s=1}^{3} \Big(\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{irs} \Big) \frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} \frac{\partial R_r}{\partial u} \frac{\partial R_s}{\partial v} = \sum_{j,k,r,s=1}^{3} \big(\delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr} \big) \frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} \frac{\partial R_r}{\partial u} \frac{\partial R_s}{\partial v} \\
&= \sum_{j,k=1}^{3} \Big[\frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} \frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} - \frac{\partial R_j}{\partial u} \frac{\partial R_k}{\partial v} \frac{\partial R_k}{\partial u} \frac{\partial R_j}{\partial v} \Big] \\
&= \Big(\sum_{j=1}^{3} \frac{\partial R_j}{\partial u} \frac{\partial R_j}{\partial u} \Big) \Big(\sum_{k=1}^{3} \frac{\partial R_k}{\partial v} \frac{\partial R_k}{\partial v} \Big) - \Big(\sum_{j=1}^{3} \frac{\partial R_j}{\partial u} \frac{\partial R_j}{\partial v} \Big) \Big(\sum_{k=1}^{3} \frac{\partial R_k}{\partial u} \frac{\partial R_k}{\partial v} \Big)\,.
\end{aligned}$$

The conclusion then follows from the fact that

$$\sum_{j=1}^{3} \frac{\partial R_j}{\partial u}\frac{\partial R_j}{\partial u} = \boldsymbol{r}_u \cdot \boldsymbol{r}_u = E\,, \quad \sum_{k=1}^{3} \frac{\partial R_k}{\partial v}\frac{\partial R_k}{\partial v} = \boldsymbol{r}_v \cdot \boldsymbol{r}_v = G\,, \quad \sum_{j=1}^{3} \frac{\partial R_j}{\partial u}\frac{\partial R_j}{\partial v} = \boldsymbol{r}_u \cdot \boldsymbol{r}_v = F\,. \qquad \square$$

Problem 3. Let $k > 0$ be a constant. Find the surface area of the cone $z = k\sqrt{x^2+y^2}$ that lies above the region $R = \{(x,y)\,|\,x^2+y^2 \leqslant 2y\}$ in the xy-plane by the following methods:

(1) Use the formula $\displaystyle\iint_R \sqrt{1+\|(\nabla f)(x,y)\|^2}\,dA$ directly.

(2) Find a parametrization of the cone above using r,θ (from the polar coordinate) as the parameters and make use of the formula $\displaystyle\iint_D \big\|(\boldsymbol{r}_r \times \boldsymbol{r}_\theta)(r,\theta)\big\|\,d(r,\theta)$.

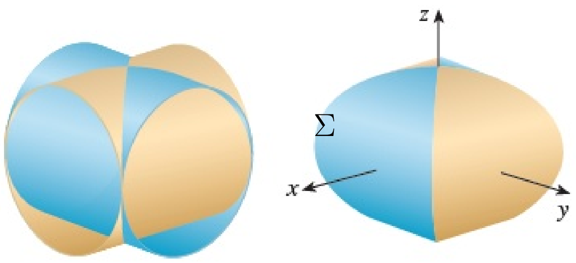
(3) Find a parametrization of the cone above using ρ,θ (from the spherical coordinate) as the parameters and make use of the formula $\displaystyle\iint_D \big\|(\boldsymbol{r}_\rho \times \boldsymbol{r}_\theta)(\rho,\theta)\big\|\,d(\rho,\theta)$.

Problem 4. Let Σ be the surface formed by rotating the curve

$$C = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x = \cos z, y = 0, -\frac{\pi}{2} \leqslant z \leqslant \frac{\pi}{2}\right\}$$

about the z-axis. Find a parametrization for Σ and compute its surface area.

Problem 5. The figure below shows the surface created when the cylinder $y^2+z^2 = 1$ intersects the cylinder $x^2+z^2 = 1$. Let Σ be the part shown in the figure.



(1) Find the area of Σ using the formula $\displaystyle\iint_R \sqrt{1+\|(\nabla f)(x,y)\|^2}\,dA$.

(2) Parameterize Σ using θ, z as parameters (from the cylindrical coordinate) and find the area of this surface using the formula $\displaystyle\iint_D \big\|(\boldsymbol{r}_\theta \times \boldsymbol{r}_z)(\theta,z)\big\|\,d(\theta,z)$.

(3) Parameterize Σ using θ, ϕ as parameters (from the spherical coordinate) and find the area of this surface using the formula $\displaystyle\iint_D \big\|(\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi)(\theta,\phi)\big\|\,d(\theta,\phi)$.

(4) Find the volume of this intersection using triple integrals.

Proof. Note that the intersection of the blue ($x^2+z^2 = 1$) and the brown ($y^2+z^2 = 1$) occurs at $x = \pm y$.

(1) In this case we view the upper part of Σ as the graph of the function $z = f(x,y) = \sqrt{1-x^2}$ on the set $R = \{(x,y)\,|\,0 \leqslant x \leqslant 1, -x \leqslant y \leqslant x\}$. Since

$$(\nabla f)(x,y) = \Big(\frac{-x}{\sqrt{1-x^2}}, 0\Big)\,,$$

using the formula for the surface area we obtain that the surface area of Σ is

$$2\iint\limits_R \sqrt{1+\|(\nabla f)(x,y)\|^2}dA = 2\int_0^1 \Big(\int_{-x}^x \sqrt{1+\frac{x^2}{1-x^2}}\,dy\Big)\,dx = 2\int_0^1 \Big(\int_{-x}^x \frac{1}{\sqrt{1-x^2}}\,dy\Big)\,dx$$
$$= 4\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = -4\sqrt{1-x^2}\Big|_{x=0}^{x=1} = 4\,.$$

(2) Introduce the cylindrical coordinate (r,θ,y), where (r,θ) is the polar coordinate on xz-plane (θ is a angle between the position (on xz-plane) vector and the x-axis). Then Σ can be parameterized by

$$\Sigma = \{\boldsymbol{r}\,|\,\boldsymbol{r}(\theta,y) = \cos\theta\boldsymbol{i} + y\boldsymbol{j} + \sin\theta\boldsymbol{k}, (\theta,y) \in \mathrm{D}\}\,,$$

where $\mathrm{D} = \Big\{(\theta,y) \in [-\pi,\pi]\times\mathbb{R}\,\Big|\, -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}, -\cos\theta \leqslant y \leqslant \cos\theta\Big\}$ (where the range of y for given θ is obtained by that $-\sqrt{1-z^2} \leqslant y \leqslant \sqrt{1-z^2}$). Since

$$\boldsymbol{r}_\theta(\theta,y) = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{k} \qquad \text{and} \qquad \boldsymbol{r}_y(\theta,y) = \boldsymbol{j}\,,$$

we have $(\boldsymbol{r}_\theta \times \boldsymbol{r}_y)(\theta,y) = -\cos\theta\boldsymbol{i} - \sin\theta\boldsymbol{k}$; thus using the formula for the surface area of parametric surfaces we obtain that the surface area of Σ is

$$\iint\limits_{\mathrm{D}} \|(\boldsymbol{r}_\theta \times \boldsymbol{r}_y)(\theta,y)\|d(\theta,y) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\cos\theta}^{\cos\theta} dyd\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta = 4\,.$$

(3) Introduce the spherical coordinate (ρ,θ,ϕ), where θ is the angle in (2) and $\frac{\pi}{2} - \phi$ is the angle between the position vector (in space) and the y-axis. Then a point (x,y,z) in space can be expressed as

$$x = \rho\cos\theta\cos\phi\,, \quad y = \rho\sin\phi\,, \quad z = \rho\sin\theta\cos\phi \quad \theta \in [-\pi,\pi], \phi \in [-\pi/2,\pi/2]\,.$$

For a fixed θ and ϕ, a point on Σ satisfies

$$x^2 + z^2 = 1 \quad \Leftrightarrow \quad \rho^2\cos^2\phi = 1 \quad \Leftrightarrow \quad \rho = \frac{1}{\cos\phi} = \sec\phi\,.$$

Therefore, Σ can be parameterized by

$$\Sigma = \{\boldsymbol{r}\,|\,\boldsymbol{r}(\theta,\phi) = \cos\theta\boldsymbol{i} + \tan\phi\boldsymbol{j} + \sin\theta\boldsymbol{k}, (\theta,y) \in \mathrm{D}\}\,,$$

where $\mathrm{D} = \Big\{(\theta,\phi) \in [-\pi,\pi]\times[-\pi/2,\pi/2]\,\Big|\, -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}, -\arctan\cos\theta \leqslant \phi \leqslant \arctan\cos\theta\Big\}$. Since

$$\boldsymbol{r}_\theta(\theta,\phi) = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{k} \qquad \text{and} \qquad r_\phi(\theta,\phi) = \sec^2\phi\boldsymbol{j}\,,$$

we have $(\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi)(\theta,\phi) = -\sec^2\phi\big(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{k}\big)$; thus using the formula for the surface area of parametric surfaces we obtain that the surface area of Σ is

$$\iint\limits_{\mathrm{D}} \frac{1}{\sin^2\phi}\,d(\theta,\phi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\arctan\cos\theta}^{\arctan\cos\theta} \sec^2\phi\,d\phi d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big(\tan\phi\Big|_{\phi=-\arctan\cos\theta}^{\phi=\arctan\cos\theta}\Big)\,d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta = 4\,.$$

(4) Under the setting of (1), we find that the volume of this intersection is

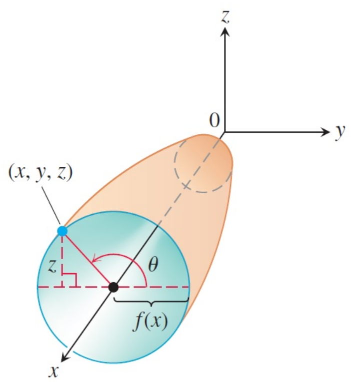
$$8\iint\limits_{R} f(x,y)\,dA = 8\int_0^1 \Big(\int_{-x}^{x} \sqrt{1-x^2}\,dy\Big)\,dx = 16\int_0^1 x\sqrt{1-x^2}\,dx$$

$$= -\frac{16}{3}(1-x^2)^{\frac{3}{2}}\Big|_{x=0}^{x=1} = \frac{16}{3}\,.$$

□

Problem 6. Let Σ be the surface obtained by rotating the smooth curve $y = f(x)$, $a \leqslant x \leqslant b$ about the x-axis, where $f(x) > 0$.

1. Show that
$$\boldsymbol{r}(x,\theta) = x\boldsymbol{i} + f(x)\cos\theta\boldsymbol{j} + f(x)\sin\theta\boldsymbol{k}, \quad (x,\theta) \in [a,b]\times[0,2\pi]\,,$$
is a parametrization of Σ, where θ is the angle of rotation about the x-axis (see the accompanying figure).



2. Show that the surface area of Σ is
$$\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$
using the formula $\displaystyle\iint\limits_{D} \|(\boldsymbol{r}_r \times \boldsymbol{r}_\theta)(r,\theta)\|\,d(r,\theta)$.

Proof. 2. We compute $\boldsymbol{r}_x$ and $\boldsymbol{r}_\theta$ and obtain that

$$\boldsymbol{r}_x(x,\theta) = \boldsymbol{i} + f'(x)\cos\theta\boldsymbol{j} + f'(x)\sin\theta\boldsymbol{k} \qquad\text{and}\qquad \boldsymbol{r}_\theta(x,\theta) = -f(x)\sin\theta\boldsymbol{i} + f(x)\cos\theta\boldsymbol{k}\,;$$

thus

$$(\boldsymbol{r}_x\cdot\boldsymbol{r}_x)(x,\theta) = 1 + f'(x)^2\,, \qquad (\boldsymbol{r}_x\cdot\boldsymbol{r}_\theta)(x,\theta) = 0\,, \qquad (\boldsymbol{r}_\theta\cdot\boldsymbol{r}_\theta)(x,\theta) = f(x)^2\,.$$

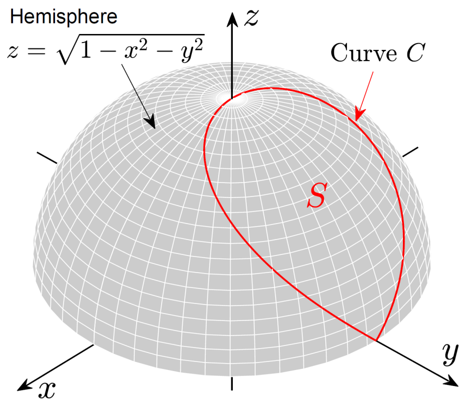
By problem 2, we have

$$\|(\boldsymbol{r}_x \times \boldsymbol{r}_\theta)(x,\theta)\|^2 = f(x)^2\big[1 + f'(x)^2\big]$$

so that using formula of the surface area of parametric surfaces we find that the surface area of Σ is

$$\iint\limits_{[a,b]\times[0,2\pi]} \|(\boldsymbol{r}_x \times \boldsymbol{r}_\theta)(x,\theta)\| d(x,\theta) = \int_a^b \Big(\int_0^{2\pi} f(x)\sqrt{1+f'(x)^2}\, d\theta dx$$
$$= 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\, dx\,. \qquad \square$$

Problem 7. Let S be the subset of the upper hemisphere $z = \sqrt{1-x^2-y^2}$ enclosed by the curve C shown in the figure below



where each point of C corresponds to some point $(\cos t \sin t, \sin^2 t, \cos t)$ with $t \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$. Find the surface of S via the following steps.

(1) The surface S can be parameterized by

$$S = \Big\{\boldsymbol{r} \,\Big|\, \boldsymbol{r} = \cos\theta\sin\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\phi\,\boldsymbol{k} \text{ for some } (\theta,\phi) \in D\Big\}\,.$$

Find the domain D inside the rectangle $[0,2\pi]\times[0,\pi]$.

(2) Find the surface area of S using the formula $\displaystyle\iint\limits_D \big\|(\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi)(\theta,\phi)\big\|\, d(\theta,\phi)$.

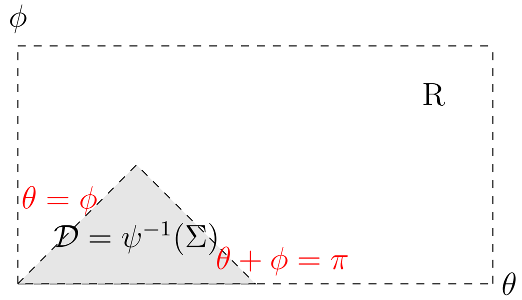
Proof. (1) Let $\mathrm{R} = (0,2\pi)\times(0,\pi)$ and $\psi : \mathrm{R} \to \mathbb{R}^3$ be defined by

$$\psi(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)\,,$$

and we would like to find a region $D \subseteq \mathrm{R}$ such that $\psi(D) = \Sigma$.

Suppose that $\gamma(t) = \big(\theta(t), \varphi(t)\big)$, $t \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]$, is a curve in R such that $(\psi\circ\gamma)(t) = \boldsymbol{r}(t)$. Then for $t \in \big[0, \frac{\pi}{2}\big]$, the identity $\cos t = \cos\phi(t)$ implies that $\phi(t) = t$; thus the identities $\cos t\sin t = \cos\theta(t)\sin\phi(t)$ and $\sin t\sin t = \sin\theta(t)\sin\phi(t)$ further imply that $\theta(t) = t$.

On the other hand, for $t \in \big[-\frac{\pi}{2}, 0\big]$, the identity $\cos t = \cos\phi(t)$, where $\phi(t) \in (0,\pi)$, implies that $\phi(t) = -t$; thus the identities $\cos t\sin t = \cos\theta(t)\sin\phi(t)$ and $\sin t\sin t = \sin\theta(t)\sin\phi(t)$ further imply that $\theta(t) = \pi + t$.



(2) First we compute $\big\|(\psi_\theta \times \psi_\phi)(\theta,\phi)\big\|$ as follows:

$$\begin{aligned}\big\|(\psi_\theta \times \psi_\phi)(\theta,\phi)\big\|^2 &= \big\|(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0) \times (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)\big\|^2 \\ &= \big\|(-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -(\sin^2\theta+\cos^2\theta)\sin\phi\cos\phi)\big\|^2 \\ &= (\cos^2\theta+\sin^2\theta)\sin^4\phi + \sin^2\phi\cos^2\phi = \sin^2\phi\,.\end{aligned}$$

Therefore, using the formula for the surface area of parametric surfaces we obtain that the surface area of S is

$$\begin{aligned}\iint\limits_D \big\|(\psi_\theta \times \psi_\phi)(\theta,\phi)\big\| d(\theta,\phi) &= \int_0^{\frac{\pi}{2}}\int_\phi^{\pi-\phi} \sin\phi\, d\theta d\phi = \int_0^{\frac{\pi}{2}} (\pi - 2\phi)\sin\phi\, d\phi \\ &= \big(-\pi\cos\phi + 2\phi\cos\phi - 2\sin\phi\big)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} = \pi - 2\,.\end{aligned}$$ ▫

Remark 0.1. Another way to parameterize S is to view S as the graph of function $z = \sqrt{1-x^2-y^2}$ over D, where D is the projection of S along z-axis onto xy-plane. We note that the boundary of D can be parameterized by

$$\widetilde{\boldsymbol{r}}(t) = (\cos t\sin t, \sin t\sin t)\,, \qquad t \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]\,.$$

Let $(x,y) \in \partial D$. Then $x^2+y^2 = y$; thus S can also be parameterized by $\psi : D \to \mathbb{R}^3$, where

$$\psi(x,y) = \big(x, y, \sqrt{1-x^2-y^2}\big) \quad \text{and} \quad D = \big\{(x,y)\,\big|\, x^2+y^2 \leqslant y\big\}\,.$$

Therefore, with f denoting the function $f(x,y) = \sqrt{1-x^2-y^2}$, the surface area of S can be computed by

$$\begin{aligned}\int_D \sqrt{1+\|(\nabla f)(x,y)\|^2}\, d\mathbb{A} &= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} \frac{1}{\sqrt{1-x^2-y^2}}\, dxdy \\ &= \int_0^1 \arcsin\frac{x}{\sqrt{1-y^2}}\Big|_{x=-\sqrt{y-y^2}}^{x=\sqrt{y-y^2}} dy = 2\int_0^1 \arcsin\frac{\sqrt{y}}{\sqrt{1+y}}\, dy\,;\end{aligned}$$

thus making a change of variable $y = \tan^2\theta$ we conclude that

$$\begin{aligned}\text{the surface area of } S &= 2\int_0^{\frac{\pi}{4}} \arcsin\frac{\tan\theta}{\sec\theta}\, d(\tan^2\theta) = 2\int_0^{\frac{\pi}{4}} \theta\, d\big(\tan^2\theta\big) \\ &= 2\Big[\theta\tan^2\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2\theta d\theta\Big] \\ &= 2\Big[\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\, d\theta\Big] = 2\Big[\frac{\pi}{4} - (\tan\theta - \theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{4}}\Big] \\ &= 2\Big[\frac{\pi}{4} - \Big(1 - \frac{\pi}{4}\Big)\Big] = \pi - 2\,.\end{aligned}$$