Exercise Problem Sets 11

May 10. 2024

Problem 1. Evaluate the following iterated integrals.

(1) $\int_{-1}^{1}\Big(\int_{0}^{1} ye^{x^2+y^2}\,dx\Big)dy$ (2) $\int_{0}^{2}\Big(\int_{y}^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\,dx\Big)dy$ (3) $\int_{0}^{1}\Big(\int_{\sqrt{y}}^{1} e^{x^3}\,dx\Big)dy$

(4) $\int_{0}^{1}\Big(\int_{y}^{1} \frac{1}{1+x^4}\,dx\Big)dy$ (5) $\int_{0}^{4}\Big(\int_{\frac{x}{2}}^{2} \sin(y^2)\,dy\Big)dx$ (6) $\int_{0}^{4}\Big(\int_{\sqrt{x}}^{2} \frac{1}{y^3+1}\,dy\Big)\,dx$

(7) $\int_{0}^{2}\Big(\int_{x}^{2} x\sqrt{1+y^3}\,dy\Big)dx$ (8) $\int_{0}^{2}\Big(\int_{\frac{y}{2}}^{1} \exp(x^2)\,dx\Big)dy$ (9) $\int_{0}^{1}\Big(\int_{0}^{1} \frac{y}{1+x^2y^2}\,dx\Big)\,dy$

(10) $\int_{0}^{\frac{\pi}{2}}\Big(\int_{x}^{\frac{\pi}{2}} \frac{\sin y}{y}\,dy\Big)dx$ (11) $\int_{0}^{2}\Big(\int_{y^2}^{4} \sqrt{x}\sin x\,dx\Big)dy$ (12) $\int_{0}^{2}\Big(\int_{0}^{4-x^2} \frac{xe^{2y}}{4-y}dy\Big)dx$

(13) $\int_{0}^{1}\Big(\int_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\,dx\Big)dy$ (14) $\int_{-5}^{5}\Big[\int_{0}^{\sqrt{25-x^2}}\Big(\int_{0}^{\frac{1}{x^2+y^2}} \sqrt{x^2+y^2}\,dz\Big)dy\Big]dx$

(15) $\int_{0}^{4}\Big[\int_{0}^{1}\Big(\int_{2y}^{y} \frac{2\cos(x^2)}{\sqrt{z}}\,dx\Big)dy\Big]dz$ (16) $\int_{0}^{1}\Big[\int_{0}^{1}\Big(\int_{x^2}^{1} xz\exp(zy^2)\,dy\Big)dx\Big]dz$

(17) $\int_{0}^{1}\Big[\int_{\sqrt[3]{z}}^{1}\Big(\int_{0}^{\ln 3} \frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,dx\Big)dy\Big]dz$ (18) $\int_{0}^{2}\Big[\int_{0}^{4-x^2}\Big(\int_{0}^{x} \frac{\sin(2z)}{4-z}\,dy\Big)dz\Big]dx$

Problem 2. Evaluate the double integral $\iint\limits_R f(x,y)\,dA$ with the following f and R.

(1) $f(x,y)=y^2e^{xy}$, and R is the region bounded by $y=x$, $y=4$ and $x=0$.

(2) $f(x,y)=xy$, and R is the region bounded by the line $y=x-1$ and parabola $y^2=2x+6$.

(3) $f(x,y)=x^2+x^2y^3-y^2\sin x$, and $R=\big\{(x,y)\,\big|\,|x|+|y|\leqslant 1\big\}$.

(4) $f(x,y)=|x|+|y|$, and $R=\big\{(x,y)\,\big|\,|x|+|y|\leqslant 1\big\}$.

(5) $f(x,y)=xy$, and R is the region in the first quadrant bounded by curves $x^2+y^2=4$, $x^2+y^2=9$, $x^2-y^2=1$ and $x^2-y^2=4$.

(6) $f(x,y)=x$, and R is the region in the first quadrant bounded by curves $4x^2-y^2=4$, $4x^2-y^2=16$, $y=x$ and the x-axis.

(7) $f(x,y)=\exp(-x^2-4y^2)$, and $R=\big\{(x,y)\,\big|\,x^2+4y^2\leqslant 1\big\}$.

(8) $f(x,y)=\exp\big(\frac{2y-x}{2x+y}\big)$, and R is the trapezoid with vertices $(0,2)$, $(1,0)$, $(4,0)$ and $(0,8)$.

Problem 3. Evaluate the integral $\int_0^2 \big[\arctan(\pi x)-\arctan x\big]\,dx$ by converting the integral into a double integral and evaluating the double integral by changing the order of integration.

Problem 4. Evaluate the integral $\int_0^1 \frac{x^b - x^a}{\ln x}\, dx$, where $0 < a < b$ are constants, by converting the integral into a double integral and evaluating the double integral by changing the order of integration.

Problem 5. Evaluate the integral $\int_0^\infty \frac{e^{-x} - e^{-3x}}{x}\, dx$ by converting the integral into a double integral and evaluating the double integral by changing the order of integration.

Problem 6. Let a, b be positive constants. Evaluate the integral

$$\int_0^a \left(\int_0^b \exp\left(\max\{b^2x^2, a^2y^2\} \right) dy \right) dx\,.$$

Problem 7. Show that if $\lambda > \frac{1}{2}$, there does not exist a real-valued continuous function u such that for all x in the closed interval $[0, 1]$,

$$u(x) = 1 + \lambda \int_x^1 u(y)u(y - x)\, dy\,.$$

Hint: Assume the contrary that there exists such a function u. Integrate the equation above on the interval $[0, 1]$.