Exercise Problem Sets 8

Apr. 19. 2024

Problem 1. Use the chain rule for functions of several variables to compute $\frac{dz}{dt}$ or $\frac{dw}{dt}$.

(1) $z = \sqrt{1+xy}$, $x = \tan t$, $y = \arctan t$.

(2) $w = x \exp\left(\frac{y}{z}\right)$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$.

(3) $w = \ln\sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$.

(4) $w = xy\cos z$, $x = t$, $y = t^2$, $z = \arccos t$.

(5) $w = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = \arctan t$, $z = e^t$.

Problem 2. Use the chain rule for functions of several variables to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(1) $z = \arctan(x^2 + y^2)$, $x = s\ln t$, $y = te^s$.

(2) $z = \arctan\frac{x}{y}$, $x = s\cos t$, $y = s\sin t$.

(3) $z = e^x \cos y$, $x = st$, $y = s^2 + t^2$.

Problem 3. Assume that $z = f\big(ts^2, \frac{s}{t}\big)$, $\frac{\partial f}{\partial x}(x,y) = xy$, $\frac{\partial f}{\partial y}(x,y) = \frac{x^2}{2}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Problem 4. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at given points.

(1) $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$, $(x,y,z) = (\pi,\pi,\pi)$.

(2) $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$, $(x,y,z) = (1, \ln 2, \ln 3)$.

(3) $z = e^x\cos(y+z)$, $(x,y,z) = (0,-1,1)$.

Problem 5. Let f be differentiable, and $z = \frac{1}{x}\big[f(x-y) + g(x+y)\big]$. Show that

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right) = x^2\frac{\partial^2 z}{\partial y^2}.$$

Problem 6. Let f be differentiable, and $z = \frac{1}{y}\big[f(ax+y) + g(ax-y)\big]$. Show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{y^2}\frac{\partial}{\partial y}\left(y^2\frac{\partial z}{\partial y}\right).$$

Problem 7. Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $z = f(x,y)$.

(1) Show that $\frac{\partial z}{\partial r} = f_x\cos\theta + f_y\sin\theta$ and $\frac{1}{r}\frac{\partial r}{\partial \theta} = -f_x\sin\theta + f_y\cos\theta$.

(2) Solve the equations in part (1) to express f_x and f_y in terms of $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$.

(3) Show that $(f_x)^2 + (f_y)^2 = \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$.

(4) Suppose in addition that f_x and f_y are differentiable. Show that

$$f_{xx} + f_{yy} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\,.$$

Problem 8. Let $f(x,y) = \sqrt[3]{xy}$.

(1) Show that f is continuous at $(0,0)$.

(2) Show that f_x and f_y exist at the origin but that the directional derivatives at the origin in all other directions do not exist.

Problem 9. Let

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

(1) Show that the directional derivative of f at the origin exists in all directions $\boldsymbol{u}$, and

$$(D_{\boldsymbol{u}} f)(0,0) = \Big(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\Big) \cdot \boldsymbol{u}\,.$$

(2) Determine whether f is differentiable at $(0,0)$ or not.

Problem 10. Let $\boldsymbol{u} = (a,b)$ be a unit vector and f be twice continuously differentiable. Show that

$$D^2_{\boldsymbol{u}} f = f_{xx} a^2 + 2 f_{xy} ab + f_{yy} b^2\,,$$

where $D^2_{\boldsymbol{u}} f = D_{\boldsymbol{u}}(D_{\boldsymbol{u}} f)$.

Problem 11. Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and that a, b are constants.

(1) $\nabla(au + bv) = a\nabla u + b\nabla v$.

(2) $\nabla(uv) = u\nabla v + v\nabla u$.

(3) $\nabla\left(\frac{u}{v}\right) = \dfrac{v\nabla u - u\nabla v}{v^2}$.

(4) $\nabla(u^n) = nu^{n-1}\nabla u$.