Exercise Problem Sets 5

Mar. 30. 2024

Problem 1. In class we have introduced the permutation symbol ε_{ijk} and use it to define the cross product: for two given vectors $\mathbf{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k} = \sum_{i=1}^{3} u_i\mathbf{e}_i$ and $\mathbf{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k} = \sum_{i=1}^{3} v_i\mathbf{e}_i$, the cross product $\mathbf{u}\times\mathbf{v}$ is defined by

$$\mathbf{u}\times\mathbf{v} = \sum_{i=1}^{3}\Big(\sum_{j,k=1}^{3}\varepsilon_{ijk}u_jv_k\Big)\mathbf{e}_i = \sum_{i,j,k=1}^{3}\varepsilon_{ijk}u_jv_k\mathbf{e}_i\,.$$

Use the summation notation above without expanding the sum (不要展開成向量和的形式，直接用 Σ 操作) and the identity

$$\sum_{i=1}^{3}\varepsilon_{ijk}\varepsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$$

to prove the following.

(1) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ for all vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ in space. (Is the associative law $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ true?)

(2) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$ for all vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ in space.

Problem 2.

(1) Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is

$$d = \frac{\|\mathbf{a}\times\mathbf{b}\|}{\|\mathbf{a}\|}\,,$$

where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.

(2) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is

$$d = \frac{\big|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\big|}{\|\mathbf{a}\times\mathbf{b}\|}\,,$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$ and $\mathbf{c} = \overrightarrow{QP}$.

Problem 3. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

Problem 4. Replace the polar equations in the following questions with equivalent Cartesian equations.

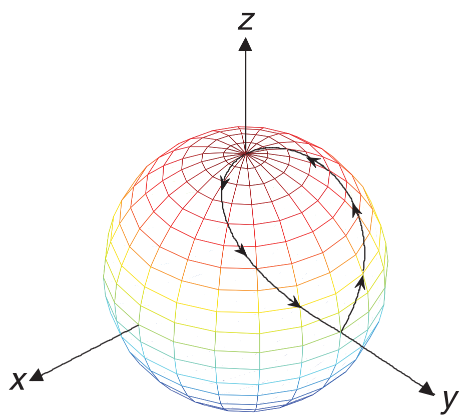
(1) $r^2\sin 2\theta = 2$ (2) $r = 4\tan\theta\sec\theta$ (3) $r = \csc\theta e^{r\cos\theta}$ (4) $r\sin\theta = \ln r + \ln\cos\theta$.

Problem 5. Let C be a smooth curve parameterized by

$$\boldsymbol{r}(t) = (\cos t \sin t, \sin t \sin t, \cos t)\,, \qquad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

(1) Show that C is a closed curve on the unit sphere $\mathbb{S}^2$.

(2) Using the spherical coordinate, the curve C above corresponds to a curve on the $\theta\phi$-plane. Find the curve in the region $\{(\theta, \phi) \,|\, 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \phi \leqslant \pi\}$.



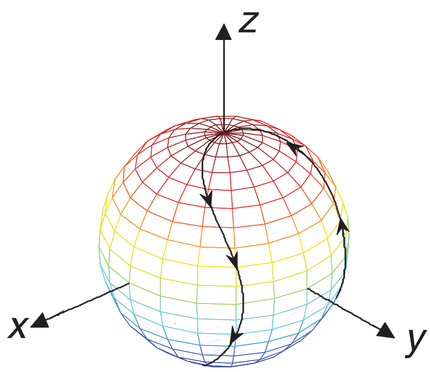
Remark: 想像球面是地球，有人開飛機飛行了 C 這個路線。這個路線在世界地圖上對應到另一個曲線，第二小題即是要求在世界地圖上這個曲線為何。

Problem 6. Let C be a smooth curve parameterized by

$$\boldsymbol{r}(t) = \big(\cos(\sin t) \sin t, \sin(\sin t) \sin t, \cos t\big)\,, \qquad t \in [0, 2\pi]\,.$$

(1) Show that C is a closed curve on the unit sphere $\mathbb{S}^2$.

(2) Using the spherical coordinate, the curve C above corresponds to a curve on the $\theta\phi$-plane. Find the curve in the region $\{(\theta, \phi) \,|\, 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \phi \leqslant \pi\}$.



Problem 7. Let C be a curve parameterized by the vector-valued function $\boldsymbol{r} : [0, 1] \to \mathbb{R}^2$,

$$\boldsymbol{r}(t) = \left(\frac{e^t - e^{-t}}{e^t + e^{-t}}, \frac{2}{e^t + e^{-t}}\right), \quad 0 \leqslant t \leqslant 1\,.$$

(1) Show that C is part of the unit circle centered at the origin.

(2) Plot the curve C. (The plot does not have to be very precise. You only need to specify the starting and end points as well as the orientation.)

(3) Find the length of the curve C.

Problem 8. Let C be the curve given by the parametric equations

$$x(t) = \frac{3+t^2}{1+t^2}\,, \qquad y(t) = \frac{2t}{1+t^2}$$

on the interval $t \in [0,1]$.

(1) In fact C is the graph of a function $y = f(x)$. Find f.

(2) Find the arc length of the curve C.

Problem 9. Parametrize the curve

$$\mathbf{r} = \mathbf{r}(t) = \arctan\frac{t}{\sqrt{1-t^2}}\boldsymbol{i} + \arcsin t\boldsymbol{j} + \arccos t\boldsymbol{k}, \quad t \in \Big[-1, 0.5\Big],$$

in the same orientation in terms of arc-length measured from the point where $t = 0$.

Problem 10. Parametrize the curve

$$\mathbf{r} = \mathbf{r}(t) = \arcsin\frac{t}{\sqrt{1+t^2}}\boldsymbol{i} + \arctan t\boldsymbol{j} + \arccos\frac{1}{\sqrt{1+t^2}}\boldsymbol{k}, \quad t \in [-1,1],$$

in the same orientation in terms of arc-length measured from the point where $t = 0$.

Problem 11. Let C_1 be the polar graph of the polar function $r = 1 + \cos\theta$ (which is a cardioid), and C_2 be the polar graph of the polar function $r = 3\cos\theta$ (which is a circle). See the following figure for reference.

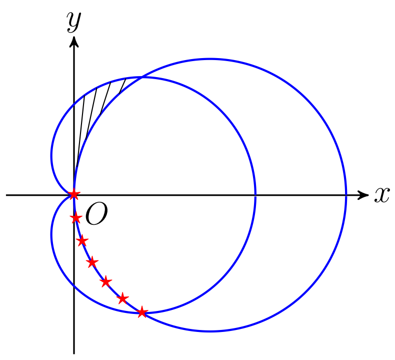


Figure 1: The polar graphs of the polar equations $r = 1 + \cos\theta$ and $r = 3\cos\theta$

(1) Find the intersection points of C_1 and C_2.

(2) Find the line L passing through the lowest intersection point and tangent to the curve C_2.

(3) Identify the curve marked by $\star$ on the θr-plane for $0 \leqslant \theta \leqslant 2\pi$.

(4) Find the area of the shaded region.

Problem 12. Let R be the region bounded by the lemniscate $r^2 = 2\cos 2\theta$ and is outside the circle $r = 1$ (see the shaded region in the graph).

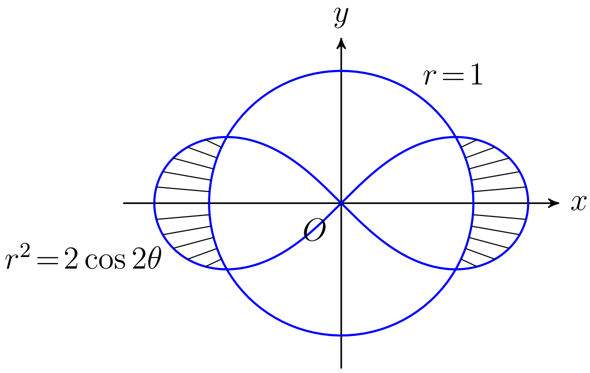


Figure 2: The polar graphs of the polar equations $r^2 = 2\cos 2\theta$ and $r = 1$

(1) Find the area of R.

(2) Find the slope of the tangent line passing thought the point on the lemniscate corresponding to $\theta = \dfrac{\pi}{6}$.

(3) Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

(a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{0.1}$$

for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) Solving (0.1), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1+4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = 2 \times \left[\pi \int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} c(x)dx - \pi \int_{\frac{\sqrt{3}}{2}}^{1} d(x)dx\right].$$

Compute $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{1} \left[d(x) - (1 - x^2)\right]dx$.

(c) Evaluate I by first computing the integral $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} \sqrt{1+4x^2}\,dx$, and then find I.

(4) Find the surface area of the surface of revolution obtained by rotating the boundary of R about the x-axis.

Problem 13. Let R be the region bounded by the circle $r = 1$ and outside the lemniscate $r^2 = -2\cos 2\theta$, and is located on the right half plane (see the shaded region in the graph).

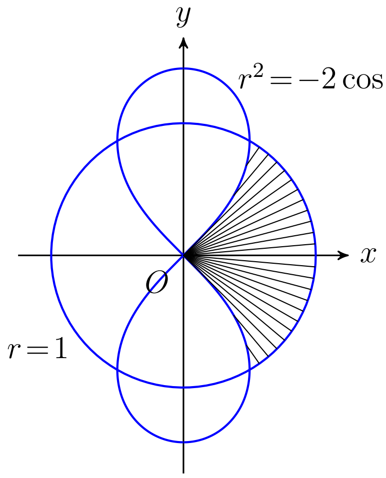


Figure 3: The polar graphs of the polar equations $r = 1$ and $r^2 = -2\cos 2\theta$

(1) Find the points of intersection of the circle $r = 1$ and the lemniscate $r^2 = -2\cos 2\theta$.

(2) Show that the straight line $x = \dfrac{1}{2}$ is tangent to the lemniscate at the points of intersection on the right half plane.

(3) Find the area of R.

(4) Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

(a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{0.2}$$

for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) Solving (0.2), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1-4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = \pi \int_0^{\frac{1}{2}} c(x)dx + \pi \int_{\frac{1}{2}}^1 d(x)dx.$$

Compute $\displaystyle\int_{\frac{1}{2}}^1 \Big[d(x) - (1 - x^2)\Big]dx$.

(c) Evaluate I by first computing the integral $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1-4x^2}dx$, and then find I.

(5) Find the area of the surface of revolution obtained by rotating the boundary of R about the x-axis.