Exercise Problem Sets 4

Mar. 15. 2024

Problem 1. Find the interval of convergence of the following power series.

(1) $\sum_{n=1}^{\infty}\big(1+\frac{1}{n}\big)^n x^n$ (2) $\sum_{n=1}^{\infty}(\ln n)x^n$ (3) $\sum_{n=1}^{\infty}\big(\sqrt{n+1}-\sqrt{n}\big)x^n$ (4) $\sum_{n=1}^{\infty}\big(\frac{n}{n+1}\big)^{n^2}x^n$

(5) $\sum_{n=1}^{\infty}\frac{n!}{(2n)!}x^n$ (6) $\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}x^{2n+1}$ (7) $\sum_{n=1}^{\infty}\frac{(-1)^n 3\cdot 7\cdot 11\cdot\cdots\cdot(4n-1)}{4^n}x^n$

(8) $\sum_{n=1}^{\infty}\frac{1}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}x^n$ (10) $\sum_{n=1}^{\infty}\frac{1}{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}x^n$ (9) $\sum_{n=1}^{\infty}\frac{n!}{3\cdot 6\cdot 9\cdot\cdots\cdot(3n)}x^n$

(10) $\sum_{n=1}^{\infty}\frac{k(k+1)(k+2)\cdots(k+n-1)}{n!}x^n$, where k is a positive integer;

(11) $\sum_{n=0}^{\infty}\frac{(n!)^k}{(kn)!}x^n$, where k is a positive integer; (12) $\sum_{n=2}^{\infty}\frac{x^n}{n\ln n}$ (13) $\sum_{n=2}^{\infty}\frac{x^n}{n(\ln n)^2}$

(14) $\sum_{n=1}^{\infty}\big[2+(-1)^n\big](x+1)^{n-1}$

Problem 2. The function J_0 defined by

$$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is called the Bessel function of the first kind of order 0. Find its domain (that is, the interval of convergence).

Problem 3. The function J_1 defined by

$$J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

is called the Bessel function of the first kind of order 1. Find its domain (that is, the interval of convergence).

Problem 4. The function A defined by

$$A(x)=1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\cdots$$

is called an Airy function after the English mathematician and astronomer Sir George Airy (1801–1892). Find the domain of the Airy function.

Solution. Write

$$A(x)=\sum_{k=0}^{\infty}a_k x^k=\sum_{k=0}^{\infty}a_{3k}x^{3k}\,.$$

Then a_k satisfies

$$a_{3k+3} = a_{3k}\frac{(3k+1)}{(3k+2)(3k+3)} \quad \forall\, k \in \mathbb{N}\cup\{0\} \qquad \text{and} \qquad a_0 = 1\,.$$

Therefore, for all $x \in \mathbb{R}$,

$$\lim_{k\to\infty}\left|\frac{a_{3(k+1)}x^{3(k+1)}}{a_{3k}x^{3k}}\right| = \lim_{k\to\infty}\frac{(3k+1)}{(3k+2)(3k+3)}|x|^3 = 0\,;$$

thus the radius of convergence of the Airy function is ∞. □

Problem 5. A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots\,;$$

that is, its coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geqslant 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

Proof. Write $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, where $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geqslant 0$. Then the fact that $\lim\limits_{n\to\infty} 2^{\frac{1}{n}} = 1$ implies that

$$\lim_{n\to\infty}\sqrt[n]{|c_n||x|^n} = |x|\,;$$

thus we conclude that the series converges for $|x| < 1$ and diverges if $|x| > 1$. Therefore, the radius of convergence is 1.

Clearly the power series does not converges at $x = \pm 1$ by the n-th term's test, so the interval of convergence of the power series is $(-1, 1)$. □

Problem 6. To find the sum of the series $\sum\limits_{n=1}^{\infty}\frac{n^2}{2^n}$, express $\frac{1}{1-x}$ as a geometric series, differentiate both sides of the resulting equation with respect to x, multiply both sides of the result by x, differentiate again, multiply by x again, and set x equal to $\frac{1}{2}$. What do you get?

Problem 7. Complete the following.

(1) Use the power series of $y = \arctan x$ to show that

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n}$$

(2) Using $x^3 + 1 = (x+1)(x^2 - x + 1)$, rewrite the integral $\int_0^{\frac{1}{2}}\frac{dx}{x^2 - x + 1}$ and then express $\frac{1}{1+x^3}$ as the sum of a power series to prove the following formula for π:

$$\pi = \frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\Big(\frac{2}{3n+1} + \frac{1}{3n+2}\Big)\,.$$

Problem 8. Show that the Bessel function of the first kind of order 0, denoted by J_0 and defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2},$$

satisfies the differential equation

$$x^2 y''(x) + xy'(x) + x^2 y(x) = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

Problem 9. Show that the Bessel function of the first kind of order 1, denoted by J_1 and defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}},$$

satisfies the differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - 1)y(x) = 0, \qquad y(0) = 0,\ y'(0) = \frac{1}{2}.$$

Problem 10. Suppose that $x_1(t)$ and $x_2(t)$ are functions of t satisfying the following equations

$$\begin{aligned} x_1''(t) - x_1(t) &= 0, \qquad x_1(0) = 1, \quad x_1'(0) = 0, \\ x_2''(t) - x_2(t) &= 0, \qquad x_2(0) = 0, \quad x_2'(0) = 1, \end{aligned}$$

where $'$ denotes the derivatives with respect to t.

1. Assume that the function $x_1(t)$ and $x_2(t)$ can be written as a power series (on a certain interval), that is, $x_1(t) = \sum\limits_{k=0}^{\infty} a_k t^k$ and $x_2(t) = \sum\limits_{k=0}^{\infty} b_k t^k$. Show that

$$(k+2)(k+1)a_{k+2} = a_k \ \text{ and } \ (k+2)(k+1)b_{k+2} = b_k \qquad \forall\, k \geqslant 0.$$

2. Find a_k and b_k, and conclude that x_1 and x_2 are some functions that we have seen before.

3. Find a function $x(t)$ satisfying

$$x''(t) - x(t) = 0, \qquad x(0) = a, \quad x'(0) = b.$$

Note that x can be written as the linear combination of x_1 and x_2.

Problem 11. Suppose that $x(t)$ is a function of t satisfying the following equations

$$x''(t) - 2x'(t) + 2x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1,$$

where $'$ denotes the derivatives with respect to t.

1. Assume that the function $x(t)$ can be written as a power series (on a certain interval), that is, $x(t) = \sum\limits_{k=0}^{\infty} a_k t^k$. Find $a_0, a_1, \cdots, a_5$.

2. Show that the 5-th Maclaurin polynomial of $e^t \sin t$ agrees with the 5-th Maclaurin polynomial of $x(t)$.

Problem 12. Find the power series solution to the differential equation

$$y''(x) + x^2 y(x) = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$$

What is the radius of convergence of this series solution?

Problem 13. In this problem we try to establish the following theorem

> Let the radius of convergence of the power series $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ be r for some $r > 0$.
>
> 1. If $\sum_{k=0}^{\infty} a_k r^k$ converges, then $\lim_{x \to (c+r)^-} f(x) = f(c+r)$.
> 2. If $\sum_{k=0}^{\infty} a_k (-r)^k$ converges, then $\lim_{x \to (c-r)^+} f(x) = f(c-r)$.

Prove case 1 of the theorem above through the following steps.

1. Let $A = \sum_{k=0}^{\infty} a_k r^k$, and define

$$g(x) = f(rx+c) - A = -\sum_{k=1}^{\infty} a_k r^k + \sum_{k=1}^{\infty} a_k r^k x^k = \sum_{k=0}^{\infty} b_k x^k\,,$$

where $b_k = a_k r^k$ for each $k \in \mathbb{N}$ and $b_0 = -\sum_{k=1}^{\infty} a_k r^k$. Show that the radius of convergence of g is 1 and $\sum_{k=0}^{\infty} b_k = 0$. Moreover, show that f is continuous at $c+r$ if and only if g is continuous at 1.

2. Let $s_n = b_0 + b_1 + \cdots + b_n$ and $S_n(x) = b_0 + b_1 x + \cdots + b_n x^n$. Show that

$$S_n(x) = (1-x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n$$

and conclude that

$$g(x) = \lim_{n \to \infty} S_n(x) = (1-x) \sum_{k=0}^{\infty} s_k x^k\,. \tag{0.1}$$

3. Use (0.1) to show that g is continuous at 1. Note that you might need to use ε-δ argument.

Proof. 2. Let $s_n = b_0 + b_1 + \cdots + b_n$ and $S_n(x) = b_0 + b_1 x + \cdots + b_n x^n$. Then $b_k = s_k - s_{k-1}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} S_n(x) &= \sum_{k=0}^{n} b_k x^k = b_0 + \sum_{k=1}^{n} b_k x^k = b_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k = b_0 + \sum_{k=1}^{n} s_k x^k - \sum_{k=1}^{n} s_{k-1} x^k \\ &= \sum_{k=0}^{n} s_k x^k - \sum_{k=0}^{n-1} s_k x^{k+1} = s_n x^n + \sum_{k=0}^{n-1} s_k (x^k - x^{k+1}) \\ &= s_n x^n + (1-x) \sum_{k=0}^{n-1} s_k x^k\,. \end{aligned}$$

By the fact that $\lim\limits_{n\to\infty} s_n = 0$ we have $\lim\limits_{n\to\infty} s_n x^n = 0$ for $|x| \leqslant 1$; thus

$$g(x) = \begin{cases} (1-x)\sum\limits_{k=0}^{\infty} s_k x^k & \text{if } x \in (-1,1)\,, \\ 0 & \text{if } x = 1\,. \end{cases}$$

3. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} s_n = 0$, there exists $N > 0$ such that $|s_n| < \frac{\varepsilon}{2}$ for all $n \geqslant N$. Choose $0 < \delta < 1$ such that $\delta \sum\limits_{k=0}^{N-1} |s_k| < \frac{\varepsilon}{2}$. Then if $1-\delta < x < 1$,

$$\begin{aligned} \big|g(x)\big| &\leqslant |1-x| \sum_{k=0}^{N-1} |s_k||x|^k + |1-x| \sum_{k=N}^{\infty} |s_k||x|^k \\ &\leqslant \delta \sum_{k=0}^{N-1} |s_k| + \frac{\varepsilon}{2}|1-x||x|^N \sum_{k=0}^{\infty} |x|^k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}|1-x|\frac{1}{1-|x|} = \varepsilon\,. \end{aligned}$$

Therefore, $\lim\limits_{x\to 1^-} g(x) = 0 = g(1)$ which shows that g is continuous at 1. □

Problem 14. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequence of real numbers such that the series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ both converge. Define $c_k = \sum\limits_{j=0}^{k} a_j b_{k-j}$ and $C_n = \sum\limits_{i=0}^{n} c_i$.

1. Show that if $\sum\limits_{n=0}^{\infty} a_n$ converges absolutely, then

$$\lim_{n\to\infty} C_n = \Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big) \tag{0.2}$$

by completing the following.

(a) Show that $C_n = \sum\limits_{k=0}^{n} a_{n-k} B_k$, where $B_k = \sum\limits_{i=0}^{k} b_i$ is the k-th partial sum of the series $\sum\limits_{i=0}^{\infty} b_i$.

(b) Let $A_k = \sum\limits_{i=0}^{k} a_i$ be the k-th partial sum of the series $\sum\limits_{i=0}^{\infty} a_i$, and $A = \lim\limits_{n\to\infty} A_n$, $B = \lim\limits_{n\to\infty} B_n$. Then

$$C_n - AB = \sum_{k=0}^{n} a_{n-k}(B_k - B) + (A_n - A)B\,.$$

Use the ε-N argument to show that $\lim\limits_{n\to\infty} C_n = AB$.

2. $\sum\limits_{n=0}^{\infty} c_n$ is called the Cauchy product of the series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$. Show that (0.2) may fail if both $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ converges conditionally by looking at the example $a_n = b_n = \dfrac{(-1)^n}{\sqrt{n+1}}$ for all $n \in \mathbb{N}$.

Proof. 1(a). By the definition of C_n,

$$C_n = \sum_{k=0}^{n} c_k = \sum_{k=0}^{n}\sum_{j=0}^{k} a_j b_{k-j} = \sum_{j=0}^{n}\sum_{k=j}^{n} a_j b_{k-j} = \sum_{j=0}^{n} a_j \Big(\sum_{k=j}^{n} b_{k-j} \Big)$$
$$= \sum_{j=0}^{n} a_j \Big(\sum_{\ell=0}^{n-j} b_\ell \Big) = \sum_{j=0}^{n} a_j B_{n-j} = \sum_{k=0}^{n} a_{n-k} B_k \,.$$

1(b). Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} B_n = B$, there exists $K > 0$ such that

$$|B_n - B| < \frac{\varepsilon}{3\sum\limits_{n=0}^{\infty} |a_n| + 3} \qquad \forall\, n \geqslant K \,.$$

Moreover, the n-term's test implies that $\lim\limits_{n\to\infty} a_n = 0$; thus there exists $L > 0$ such that

$$|a_n| < \frac{\varepsilon}{3\sum\limits_{k=0}^{K-1} |B_k - B| + 3} \qquad \forall\, n \geqslant L \,.$$

Finally, by the fact that $A_n \to A$ as $n \to \infty$, there exists $M > 0$ such that

$$|A_n - A| < \frac{\varepsilon}{3(|B|+1)} \qquad \forall\, n \geqslant M \,.$$

Let $N = \max\{K+L, M\}$. Then if $n \geqslant N$,

$$\begin{aligned} |C_n - AB| &\leqslant \sum_{k=0}^{K-1} |a_{n-k}||B_k - B| + \sum_{k=K}^{n} |a_{n-k}||B_k - B| + |A_n - A||B| \\ &\leqslant \frac{\varepsilon}{3\sum\limits_{k=0}^{K-1} |B_k - B| + 3} \sum_{k=0}^{K-1} |B_k - B| + \sum_{k=K}^{n} |a_{n-k}| \frac{\varepsilon}{3\sum\limits_{n=0}^{\infty} |a_n| + 3} + \frac{|B|\varepsilon}{3(|B|+1)} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \,. \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} C_n = AB$. $\square$