Exercise Problem Sets 2

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Problem 1. Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent. If it is convergent, find its sum.

(1) $a_n = \dfrac{1}{1+(\frac{2}{3})^n}$ (2) $a_n = \ln\big(\dfrac{n^2+1}{2n^2+1}\big)$ (3) $a_n = e^{-n} + \dfrac{1}{n(n+1)}$ (4) $a_n = \dfrac{1}{n^3-n}$

(5) $a_n = \dfrac{40n}{(2n-1)^2(2n+1)^2}$

Problem 2. Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

(1) $a_n = \dfrac{1}{n^{1+\frac{1}{n}}}$ (2) $a_n = \ln\big(1+\dfrac{1}{n^2}\big)$ (3) $a_n = \dfrac{2^n+3^n}{3^n+4^n}$ (4) $a_n = \tan\dfrac{1}{n}$

(5) $a_n = \sin^n \dfrac{1}{\sqrt{n}}$ (6) $a_n = \dfrac{\arctan n}{n^{1.1}}$ (7) $a_n = \Big[-\ln\big(e^2+\dfrac{1}{n^2}\big)\Big]^n$ (8) $a_n = \big(1-\dfrac{1}{n}\big)^{n^2}$

(9) $a_n = \big(1+\dfrac{1}{n}\big)^{-n^2}$ (10) $a_n = \dfrac{(n!)^2}{(2n)!}$ (11) $a_n = \dfrac{n!\ln n}{n(n+2)!}$ (12) $a_n = \dfrac{n!}{n^n}$

(13) $a_n = \dfrac{(-1)^n(3n)!}{n!(n+1)!(n+2)!}$ (14) $a_n = \dfrac{(-1)^n(n!)^n}{n^{n^2}}$ (15) $a_n = \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}$

(16) $a_n = (-1)^n\big(\sqrt{n+\sqrt{n}}-\sqrt{n}\big)$ (17) $a_n = (-1)^n\dfrac{(n!)^2 3^n}{(2n+1)!}$

Problem 3. Find values of x for which the following series converges.

(1) $\sum_{n=1}^{\infty}(-4)^n(x-5)^n$ (2) $\sum_{n=1}^{\infty}\dfrac{2^n}{x^n}$ (3) $\sum_{n=1}^{\infty}\dfrac{\sin^n x}{3^n}$ (4) $\sum_{n=1}^{\infty} e^{nx}$.

Problem 4. The Fibonacci sequence $\{f_n\}_{n=1}^{\infty}$ is a sequence defined recursively by

$$f_1 = 1\,,\quad f_2 = 1 \qquad\text{and}\qquad f_{n+2} = f_{n+1} + f_n \quad \forall\, n \in \mathbb{N}\,.$$

Show the following.

(1) $\dfrac{1}{f_{n-1}f_{n+1}} = \dfrac{1}{f_{n-1}f_n} - \dfrac{1}{f_n f_{n+1}}$ for all $n \geqslant 2$.

(2) $\sum_{n=2}^{\infty}\dfrac{1}{f_{n-1}f_{n+1}} = 1$.

(3) $\sum_{n=2}^{\infty}\dfrac{f_n}{f_{n-1}f_{n+1}} = 2$.

Problem 5. Consider the series $\sum_{n=1}^{\infty}\dfrac{n}{(n+1)!}$.

(1) Find the partial sum S_1, S_2, S_3 and S_4. Do you recognize the denominators? Use the pattern to guess a formula for S_n.

(2) Prove your guess by induction.

(3) Show that the given series is convergent, and find the sum.

Problem 6. Find all p and q such that $\sum_{k=2}^{\infty} \frac{(\ln k)^q}{k^p}$ converges.

Problem 7. Show that if $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive terms, then $\sum_{k=1}^{\infty} \sin a_k$ converges.

Problem 8. Let $S = \sum_{k=1}^{\infty} \frac{1}{k^2}$. Euler found that $S = \frac{\pi^2}{6}$ in 1735 AD.

(1) Show that $S = 1 + \sum_{k=1}^{\infty} \frac{1}{n^2(n+1)}$.

(2) Which of the sums $\sum_{k=1}^{1000000} \frac{1}{k^2}$ or $1 + \sum_{k=1}^{1000} \frac{1}{k^2(k+1)}$ should give a better approximation of S? Explain your answer.

Hint: (1) $\frac{1}{n^2(n+1)} = \frac{1}{n^2} - \frac{1}{n(n+1)}$.

Problem 9. Find all real numbers x such that $\sum_{k=1}^{\infty} \frac{\cos(kx)}{\ln k}$ converges.

Problem 10. Show by example that $\sum_{k=1}^{\infty} a_k b_k$ may diverge even if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge.

Problem 11. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n, b_n > 0$ for all $n \geqslant N$. Define

$$c_n = b_n - b_{n+1}\frac{a_{n+1}}{a_n} \qquad \forall\, n \in \mathbb{N}\,. \tag{$\star$}$$

1. Show that if there exists a constant $r > 0$ such that $r < c_n$ for all $n \geqslant N$, then $\sum_{k=1}^{\infty} a_k$ converges.

   Hint: Rewrite ($\star$) as $b_n = c_n + \frac{a_{n+1}}{a_n} b_{n+1}$ and then obtain

$$\begin{aligned} b_N = c_N + \frac{a_{N+1}}{a_N} b_{N+1} &= c_N + \frac{a_{N+1}}{a_N}\Big(c_{N+1} + \frac{a_{N+2}}{a_{N+1}} b_{N+2}\Big) = c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} b_{N+2} \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N}\Big(c_{N+2} + \frac{a_{N+3}}{a_{N+2}} b_{N+3}\Big) = \cdots \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} c_{N+2} + \cdots + \frac{a_{N+n}}{a_N} c_{N+n} + \frac{a_{N+n+1}}{a_N} b_{N+n+1}\,. \end{aligned}$$

   Use the fact that $0 < r < c_n$ for all $n \geqslant N$ to conclude that

$$\sum_{k=N}^{N+n} a_k \leqslant \frac{a_N b_N}{r} \qquad \forall\, n \in \mathbb{N}\,.$$

   Note that then the sequence of partial sum of $\sum_{k=1}^{\infty} a_k$ then is bounded from above $\Big($by $\sum_{k=1}^{N-1} a_k + \frac{a_N b_N}{r}\Big)$.

2. Show that if $\sum_{k=1}^{\infty} \frac{1}{b_k}$ diverges and $c_n \leqslant 0$ for all $n \geqslant N$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Hint: The fact that $c_n \leqslant 0$ for all $n \geqslant N$ implies that $b_n a_n \leqslant b_{n+1} a_{n+1}$ for all $n \geqslant N$. Use this fact to conclude that

$$\frac{a_N b_N}{b_n} \leqslant a_n \qquad \forall\, n \geqslant N$$

and then apply the direct comparison test to conclude that $\sum_{k=1}^{\infty} a_k$ diverges.

Problem 12. Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms, and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. We know from class that the ratio test fails when this happens, but there are some refined results concerning this particular case.

1. **(Raabe's test)**:

   (a) If there exists a constant $\mu > 1$ such that $\frac{a_{n+1}}{a_n} < 1 - \frac{\mu}{n}$ for all $n \geqslant N$, then $\sum_{k=1}^{\infty} a_k$ converges.

   (b) If there exists a constant $0 < \mu < 1$ such that $\frac{a_{n+1}}{a_n} > 1 - \frac{\mu}{n}$ for all $n \geqslant N$, then $\sum_{k=1}^{\infty} a_k$ diverges.

   Hint: Consider the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_n = (n-1)a_n - na_{n+1}$. Then $\sum_{k=1}^{\infty} b_k$ is a telescoping series. For case (a), show that $\{na_{n+1}\}_{n=N}^{\infty}$ is a positive decreasing sequence and then conclude that $\sum_{k=1}^{\infty} b_k$ converges. Note that $b_n \geqslant (\mu-1)a_n$ for all $n \geqslant N$. For case (b), show that $\{na_{n+1}\}_{n=N}^{\infty}$ is a positive increasing sequence; thus $a_n \geqslant \frac{Na_{N+1}}{n-1}$ for all $n \geqslant N+1$ which implies that $\sum_{k=1}^{\infty} a_k$ diverges.

   Remark: 注意到 (a) 說的是如果 $\{a_n\}_{n=1}^{\infty}$ 在某項之後「遞減得夠快」，那麼 $\sum_{k=1}^{\infty} a_k$ 收斂。反之，如果 $\{a_n\}_{n=1}^{\infty}$「並非遞減得那麼快」，那麼 $\sum_{k=1}^{\infty} a_k$ 發散。

2. **(Gauss's test)**: Suppose that there exist a positive constant $\epsilon > 0$, a constant μ, and a **bounded** sequence $\{R_n\}_{n=1}^{\infty}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} \qquad \text{for all } n \geqslant N\,.$$

   (a) If $\mu > 1$, then $\sum_{k=1}^{\infty} a_k$ converges. (b) If $\mu \leqslant 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

   Hint: Show that if $\mu > 1$ or $\mu < 1$, one can apply Raabe's test to conclude Gauss's test. For the case $\mu = 1$, let $b_n = (n-1)\ln(n-1)$ for $n \geqslant 2$. Using the second result of Problem 11 to show the divergence of $\sum_{k=1}^{\infty} a_k$ (by showing that c_n defined by ($\star$) is non-positive for all large enough n).

Problem 13. Complete the following.

1. Show that $\sum_{k=1}^{\infty} \left(1 - \frac{1}{\sqrt{k}}\right)^k$ converges.

2. Show that $\sum_{k=2}^{\infty} \frac{\log(k+1) - \log k}{(\log k)^2}$ converges.

3. Use Gauss's test to show that both the general harmonic series $\sum_{k=1}^{\infty} \frac{1}{ak+b}$, where $a \neq 0$, and the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverge.

4. Show that $\sum_{k=1}^{\infty} \frac{k!}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}$ converges if $\alpha > 1$ and diverges if $\alpha \leqslant 1$.

5. Test the following "hypergeometric" series for convergence or divergence:

   (a) $\sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+k-1)} = \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \cdots$.

   (b) $1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2\gamma \cdot (\gamma+1)} + \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} + \cdots$.

Problem 14. Let $\sum_{k=1}^{\infty} a_k$ be a conditionally convergent series. Show that $\sum_{k=1}^{\infty} \big[1 + \text{sgn}(a_k)\big] a_k$ and $\sum_{k=1}^{\infty} \big[1 - \text{sgn}(a_k)\big] a_k$ both diverge. Here the sign function sgn is defined by

$$\text{sgn}(a) = \begin{cases} 1 & \text{if } a > 0\,, \\ 0 & \text{if } a = 0\,, \\ -1 & \text{if } a < 0\,. \end{cases}$$

Problem 15. A permutation of a non-empty set A is a one-to-one function from A onto A. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$.

1. Suppose that $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers. Show that $\{a_{\pi(n)}\}_{n=1}^{\infty}$ is also convergent; that is, show that if $\{b_n\}_{n=1}^{\infty}$ is a sequence defined by $b_n = a_{\pi(n)}$, then $\{b_n\}_{n=1}^{\infty}$ also converges.

2. Suppose that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. Show that $\sum_{k=1}^{\infty} a_{\pi(k)}$ is also absolutely convergent, and
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\pi(k)}\,.$$

3. Suppose that $\sum_{k=1}^{\infty} a_k$ is conditionally convergent. Show that for each $r \in \mathbb{R}$, there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that
$$\sum_{k=1}^{\infty} a_{\pi(k)} = r\,.$$