Exercise Problem Sets 1

Feb. 24. 2024

Problem 1. Determine whether the sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If it converges, find the limit.

(1) $a_n = \dfrac{\ln n}{\ln(2n)}$ (2) $a_n = \dfrac{(-1)^{n+1}n}{n+\sqrt{n}}$ (3) $a_n = n\sin\dfrac{1}{n}$ (4) $a_n = n - \sqrt{n+1}\sqrt{n+3}$

(5) $a_n = \sqrt[n]{n^2+n}$ (6) $a_n = (3^n+5^n)^{\frac{1}{n}}$ (7) $a_n = \dfrac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$

(8) $a_n = \sqrt{n}\ln\left(1+\dfrac{1}{n}\right)$ (9) $a_n = \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}$ (10) $a_n = \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n (n+1)!}$.

Problem 2. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers.

(1) Show that if $\lim\limits_{n\to\infty}(a_n+b_n)$ D.N.E. and $\lim\limits_{n\to\infty} b_n$ converges, then $\lim\limits_{n\to\infty} a_n$ D.N.E.

(2) Show that if $\sum\limits_{n=1}^{\infty}(a_n+b_n)$ diverges and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Problem 3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of real numbers defined by

$$\sigma_n = \frac{a_1+a_2+\cdots+a_n}{n} = \frac{1}{n}\sum_{k=1}^{n} a_k\,.$$

(1) Show that if $\lim\limits_{n\to\infty} a_n = a$ exists, then $\lim\limits_{n\to\infty}\sigma_n = a$.

(2) Suppose that $\lim\limits_{n\to\infty}\sigma_n = a$ exists, is it necessary that $\lim\limits_{n\to\infty} a_n = a$?

Problem 4. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers defined recursively by

$$a_{n+1} = \sqrt{2a_n} \quad \forall\, n\in\mathbb{N}\cup\{0\}\,, a_0 = \sqrt{2}\,.$$

Show the following.

1. Show that $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded from above by 2.
2. Show that $\{a_n\}_{n=1}^{\infty}$ converges and find the limit.

Problem 5. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers defined recursively by

$$a_{n+1} = \sqrt{2+a_n} \quad \forall\, n\in\mathbb{N}\cup\{0\}\,, a_0 = \sqrt{2}\,.$$

Show the following.

1. Show that $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded from above by 2.
2. Show that $\{a_n\}_{n=1}^{\infty}$ converges and find the limit.

Problem 6. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real number defined by the recursive relation

$$a_{n+1} = \frac{1}{2+a_n} \quad \forall\, n \geqslant 0\,, \qquad a_0 = \frac{1}{2}\,.$$

Complete the following.

(1) Show that the sequence $\{a_{2n}\}_{n=0}^{\infty}$ is a decreasing sequence; that is, $a_{2n+2} \leqslant a_{2n}$ for all $n \in \mathbb{N} \cup \{0\}$.

(2) Show that the sequence $\{a_{2n+1}\}_{n=0}^{\infty}$ is an increasing sequence; that is, $a_{2n+3} \geqslant a_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

(3) Show that $a_{2k+1} \leqslant a_{2\ell}$ for all $k, \ell \in \mathbb{N} \cup \{0\}$.

(4) Show that the two sequences $\{a_{2n}\}_{n=0}^{\infty}$ and $\{a_{2n+1}\}_{n=0}^{\infty}$ converges to the same limit.

(5) Show that $\{a_n\}_{n=0}^{\infty}$ converges.

Problem 7. In this problem you are asked to show that the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \big(1 + \frac{1}{n}\big)^n$ converges without knowing that $\lim\limits_{x\to\infty} \big(1 + \frac{1}{x}\big)^x = e$. Complete the following.

(1) Show that if $0 \leqslant a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b-a} < (n+1)b^n\,.$$

(2) Deduce that $b^n\big[(n+1)a - nb\big] < a^{n+1}$.

(3) Use $a = 1 + \dfrac{1}{n+1}$ and $b = 1 + \dfrac{1}{n}$ in (2) to show that $\{a_n\}_{n=1}^{\infty}$ is (strictly) increasing.

(4) Use $a = 1$ and $b = 1 + \dfrac{1}{2n}$ in (2) to show that $a_{2n} < 4$.

(5) Use (3) and (4) to show that $a_n < 4$.

(6) Deduce that $\{a_n\}_{n=1}^{\infty}$ converges.

Problem 8. Let a, b be positive real numbers, $a > b$. Let two sequence $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be given by the recursive relation

$$a_{n+1} = \frac{a_n + b_n}{2}\,,\ b_{n+1} = \sqrt{a_n b_n} \quad \forall\, n \in \mathbb{N}\,, \qquad a_1 = \frac{a+b}{2}\,,\ b_1 = \sqrt{ab}\,.$$

Complete the following.

(1) Show (by induction) that $a_n > a_{n+1} > b_{n+1} > b_n$ for all $n \in \mathbb{N}$.

(2) Deduce that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converges.

(3) Show that $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ both exist and are identical.