Exercise Problem Sets 5

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Problem 1. Let f, g be functions defined on an open interval, and $n \in \mathbb{N}$. Show that if the n-th derivatives of f and g exist on I, then

$$\begin{aligned}\frac{d^n}{dx^n}(fg)(x) &= f^{(n)}(x)g(x) + C^n_1 f^{(n-1)}(x)g'(x) + C^n_2 g^{(n-2)}(x)g''(x) + \cdots \\ &\quad + C^n_{n-2} f''(x)g^{(n-2)}(x) + C^n_{n-1} f'(x)g^{(n-1)}(x) + f(x)g^{(n)}(x) \\ &= \sum_{k=0}^{n} C^n_k f^{(n-k)}(x)g^{(k)}(x)\,,\end{aligned}$$

where $C^n_k = \dfrac{n!}{k!(n-k)!}$ is "n choose k".

Hint: Prove by induction.

Problem 2. 1. Let $n \in \mathbb{N}$. Show that $\sum\limits_{k=1}^{n-1} kx^{k-1} = \dfrac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}$ if $x \neq 1$.

2. Show that $\sum\limits_{k=1}^{n} k\cos(kx) = \dfrac{-1 + (2n+1)\sin\frac{x}{2}\sin(n+\frac{1}{2})x + \cos\frac{x}{2}\cos(n+\frac{1}{2})x}{4\sin^2\frac{x}{2}}$ if $x \in (-\pi, \pi)$.

Hint 1. Find the sum $\sum\limits_{k=1}^{n-1} x^k$ first and then observe that $\sum\limits_{k=1}^{n-1} kx^{k-1} = \sum\limits_{k=1}^{n-1} \dfrac{d}{dx}x^k$.

2. Find the sum $\sum\limits_{k=1}^{n} \sin(kx)$ first and then observe that $\sum\limits_{k=1}^{n} k\cos(kx) = \sum\limits_{k=1}^{n} \dfrac{d}{dx}\sin(kx)$.

Problem 3. For a fixed constant $a > 1$, consider the function $f(x) = \log_a x$. Suppose that you are given the fact that the limit

$$\lim_{h\to 0} \frac{\log_{10}(1+h)}{h} \approx 0.43429$$

exists.

1. Show that f is differentiable on $(0, \infty)$ for all $a > 1$.
2. Show that there exists $a > 1$ such that $f'(x) = \dfrac{1}{x}$ for all $x \in (0, \infty)$.

Hint: 1. Use the "change of base formula"（换底公式）for logarithm.

2. Define $g(a) = \dfrac{d}{dx}\Big|_{x=1} \log_a x$. Apply the intermediate value theorem to g.

Problem 4. Let $f(x) = a_1\sin x + a_2\sin(2x) + a_3\sin(3x) + \cdots + a_n\sin(nx)$, where $a_1, a_2, \cdots, a_n$ are real numbers and $n \in \mathbb{N}$. Show that if $|f(x)| \leqslant |\sin x|$ for all $x \in \mathbb{R}$, then

$$|a_1 + 2a_2 + 3a_3 + \cdots + na_n| \leqslant 1\,.$$

Problem 5. 1. Let $n \in \mathbb{N}$. Show that $\dfrac{d}{dx}\big[\sin^n x\cos(nx)\big] = n\sin^{n-1} x\cos(n+1)x\,.$

2. Find a similar formula for the derivative of $\cos^n x\cos(nx)$.

Problem 6. For $n, k \in \mathbb{N}$ satisfying $n \geqslant k$, let $r_k(x) = C_k^n x^k (1-x)^{n-k}$. Note that $\sum_{k=0}^{n} r_k(x) = 1$ for all $x \in \mathbb{R}$. Prove the following.

1. $\sum_{k=0}^{n} k r_k(x) = nx$.

2. $\sum_{k=0}^{n} k(k-1) r_k(x) = n(n-1)x^2$.

Hint: For each fixed $a \in \mathbb{R}$, consider the function $f_a(x) = (x+a)^n$. Find the derivative $f_a'(1-a)$ and $f_a''(1-a)$.