Exercise Problem Sets 4

Oct. 6. 2023

Problem 1. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

Problem 2. Suppose that a and b are positive constants. Show that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

Problem 3. Find all asymptotes of the graph of the function $f(x) = \dfrac{3x^3(x - \sqrt[3]{x^3 - x^2 + x})}{x^2 - 1}$.

Problem 4. Suppose that you know that the function $y = 2^x$ is continuous on $\mathbb{R}$, $2^a > 2^b$ for all $a > b$, and the two limits $\lim\limits_{x\to\infty} 2^x = \infty$, $\lim\limits_{x\to-\infty} 2^x = 0$. Find all the asymptotes of the graph of the function

$$f(x) = \frac{2^x + 2^{-x}}{2^x - 2^{-x}}\,.$$

Problem 5. Let f be a function defined on an open interval containing c. Show that f is differentiable at c if and only if there exists a real number L satisfying that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\big|f(c+h) - f(c) - Lh\big| \leqslant \varepsilon|h| \qquad \text{whenever} \quad |h| < \delta\,.$$

Problem 6. Let I be an open interval and $c \in I$. The left-hand and right-hand derivative of f at c, denoted by $f'(c^+)$ and $f'(c^-)$, respectively, are defined by

$$f'(c^+) = \lim_{h\to0^+} \frac{f(c+h) - f(c)}{h} \qquad \text{and} \qquad f'(c^-) = \lim_{h\to0^-} \frac{f(c+h) - f(c)}{h}$$

provides the limits exist.

1. Show that if f is differentiable at c if and only if $f'(c^+) = f'(c^-)$, and in either case we have $f'(c) = f'(c^+) = f'(c^-)$.

2. Let $f(x) = \begin{cases} x^2 & \text{if } x \leqslant 2\,, \\ mx + k & \text{if } x > 2\,. \end{cases}$ Find the value of m and k such that f is differentiable at 2.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(a+b) = f(a)f(b)$ for all $a, b \in \mathbb{R}$. Recall that in Problem 2 of Exercise 3 we have talked about (the non-negativity and) the continuity of f as long as f is continuous at 0. Suppose that f is differentiable at 0.

1. Show that f is differentiable on $\mathbb{R}$.

2. Find $f'(x)$ (in terms of f and $f'(0)$).

Problem 8. Let $f : (0, \infty) \to \mathbb{R}$ be a function satisfying $f(ab) = f(a) + f(b)$ for all $a, b > 0$.

1. Show that $f(1) = 0$.

2. Suppose that f is continuous at 1, show that f is continuous on $(0, \infty)$.

3. Suppose that f is differentiable at 1. Show that f is differentiable on $(0, \infty)$. Find $f'(x)$ in terms of $f'(1)$.