Exercise Problem Sets 2

Sept. 22. 2023

Problem 1. Let f be a function defined on an open interval containing c (except possibly at c).

1. Prove that if $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} \big|f(x)\big| = |L|$.
2. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c} \big|f(x) - L\big| = 0$.

Problem 2. Let f be a function defined on an open interval containing c and $\lim_{x\to c} f(x)$ exists. Show that there exist $\delta > 0$ and $M > 0$ such that

$$\big|f(x)\big| \leqslant M \qquad \text{whenever} \quad |x - c| < \delta\,.$$

Problem 3. Let f, g be a function defined on an open interval containing c (except possibly at c), and $f(x) \leqslant g(x)$ for all $x \neq c$. Prove that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$ both exist, then $L \leqslant K$.

Problem 4. 1. Suppose that $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 3$. Find $\lim_{x\to 2} f(x)$.

2. Suppose that $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 4$. Find $\lim_{x\to 2} f(x)$.
3. Suppose that $\lim_{x\to c} \dfrac{f(x) - p(x)}{x - c} = L$ exists, where p is a polynomial function. Find $\lim_{x\to c} f(x)$.

Problem 5. In class you are given the identity $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. Compute the following limits:

1. $\lim_{x\to 0} \dfrac{\sin(x^2)}{x}$. 2. $\lim_{x\to 0} \dfrac{x \sin x}{1 - \cos x}$. 3. $\lim_{x\to 0} \dfrac{\sin(\sin(\sin x))}{x}$.
4. $\lim_{x\to 0} \dfrac{\sin(x + c) - \sin c}{x}$, where c is a real number.

Problem 6. Show that $\lim_{x\to 0^+} x^{\frac{3}{4}} \cos \dfrac{1}{x^2} = 0$ using (1) ε-δ definition and (2) the Squeeze theorem.

Problem 7. 1. Find the limits $\lim_{x\to 2^+} \dfrac{x^2 + x - 6}{|x - 2|}$ and $\lim_{x\to 2^-} \dfrac{x^2 + x - 6}{|x - 2|}$. Determine whether the limit $\lim_{x\to 2} \dfrac{x^2 + x - 6}{|x - 2|}$ exists or not.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} a + \sin(x - 2) & \text{if } x > 2\,, \\ x^2 - 3x + b & \text{if } x \leqslant 2\,. \end{cases}$$

Find the relation between a and b so that the limit $\lim_{x\to 2} f(x)$ exists.

3. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 1 + \sin(x-2) & \text{if } x > 2\,, \\ x^2 - 3x + 3 & \text{if } x \leqslant 2\,. \end{cases}$$

Find the limit $\lim\limits_{x \to 2} \dfrac{g(x) - 1}{x - 2}$ using the left limit and right limit criteria. You can use the limit in Problem 5.

Problem 8. True or False: Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. If f and g are functions such that $\lim\limits_{x \to c^+} g(x) = K$, $\lim\limits_{y \to K^+} f(y) = f(K)$, then

$$\lim_{x \to c^+} (f \circ g)(x) = f(K)\,.$$

How about if $x \to c^+$ and $y \to K^+$ are replaced by $x \to c^-$ and $y \to K^-$, respectively?

2. Let f, g be a function defined on an open interval containing c (except possibly at c), and $f(x) < g(x)$ for all $x \neq c$. If $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = K$ both exist, then $L < K$.