Exercise Problem Sets 15

Dec. 30. 2023

Problem 1. Find at least two ways to compute the following integrals.

1. $\displaystyle\int \frac{x-1}{x^2-4x-5}\,dx$ 2. $\displaystyle\int \frac{3x^2-2}{x^3-2x-1}\,dx$ 3. $\displaystyle\int \frac{1+4\cot x}{4-\cot x}\,dx$

4. $\displaystyle\int \frac{1}{x(x^4+1)}\,dx$ 5. $\displaystyle\int \frac{4}{\tan x-\sec x}\,dx$ 6. $\displaystyle\int \frac{2}{x^6+x}\,dx$

Problem 2. Find the following indefinite integrals using the techniques of partial fractions.

1. $\displaystyle\int \frac{x}{x^4-1}\,dx$ 2. $\displaystyle\int \frac{x}{x^4+4x^2+3}\,dx$ 3. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx$ 4. $\displaystyle\int \frac{x^3+1}{x^3-x^2}\,dx$

5. $\displaystyle\int \frac{1}{x^6+1}\,dx$ 6. $\displaystyle\int \frac{1}{(x-2)(x^2+4)}\,dx$ 7. $\displaystyle\int \frac{1}{x+4+4\sqrt{x+1}}\,dx$ 8. $\displaystyle\int \frac{1}{x\sqrt{4x+1}}\,dx$

9. $\displaystyle\int \frac{1}{x^2\sqrt{4x+1}}\,dx$ 10. $\displaystyle\int \frac{1}{x+\sqrt[3]{x}}\,dx$ 11. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,dx$ 12. $\displaystyle\int \frac{1}{e^{3x}-e^x}\,dx$

13. $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$ 14. $\displaystyle\int \frac{1}{3-2\sin x}\,dx$ 15. $\displaystyle\int \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$

Problem 3. Determine if the following improper integral converges or not.

1. $\displaystyle\int_0^\infty \frac{dx}{\sqrt[3]{x^4-x^2}}$ 2. $\displaystyle\int_1^\infty \frac{dx}{x(\ln x)^\alpha}$ 3. $\displaystyle\int_1^\infty \frac{\ln x}{x^\alpha}\,dx$ 4. $\displaystyle\int_{10}^\infty \frac{dx}{x(\ln\ln x)^\alpha}$

5. $\displaystyle\int_0^\pi \frac{dx}{\sqrt{x}+\sin x}$ 6. $\displaystyle\int_0^\pi \frac{dx}{x-\sin x}$ 7. $\displaystyle\int_0^{\ln 2} x^{-2}e^{-\frac{1}{x}}\,dx$ 8. $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

9. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{e^x-x}}$ 10. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$ 11. $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ 12. $\displaystyle\int_{-1}^1 \ln|x|\,dx.$

Problem 4. Complete the following.

1. Show that the improper integral $\displaystyle\int_0^{\frac{\pi}{2}} \ln\sin x\,dx$ converges.

2. Find the value of $\displaystyle\int_0^{\frac{\pi}{2}} \ln\sin x\,dx$.

Proof. 1. We present two proof here.

(a) Let $f(x)=-\ln\sin x$ and $g(x)=-\ln x$. Then f,g are positive on $(0,1]$. Moreover, by L'Hôspital's rule,

$$\lim_{x\to0^+}\frac{f(x)}{g(x)}=\lim_{x\to0^+}\frac{\ln\sin x}{\ln x}=\lim_{x\to0^+}\frac{\cos x/\sin x}{1/x}=\lim_{x\to0^+}\frac{x\cos x}{\sin x}=1>0\,.$$

Therefore, by the limit comparison test,

$$\int_0^1 \ln\sin x\,dx \text{ converges if and only if } \int_0^1 \ln x\,dx \textit{ converges}.$$

Now, by the fact that $\lim\limits_{x\to 0^+} x\ln x = 0$,

$$\int_0^1 \ln x\,dx = \lim_{a\to 0^+}\int_a^1 \ln x\,dx = \lim_{a\to 0^+}\big(x\ln x - x\big)\Big|_{x=a}^{x=1} = \lim_{a\to 0^+}(a-1-a\ln a) = -1\,;$$

thus $\displaystyle\int_0^1 \ln x\,dx$ converges.

(b) Note that $\dfrac{2}{\pi}x \leqslant \sin x \leqslant x$ for all $x\in\big(0,\dfrac{\pi}{2}\big)$. Since $y=\ln x$ is increasing on $(0,1)$,

$$\ln\frac{2}{\pi} + \ln x \leqslant \ln\sin x \leqslant \ln x \leqslant 0 \qquad \forall\, x\in(0,1)\,.$$

Let $f(x) = -\ln\sin x$ and $g(x) = -\ln\dfrac{2}{\pi} - \ln x$. Then $0 \leqslant f(x) \leqslant g(x)$ for all $x\in(0,1]$. Now, by the fact that $\lim\limits_{x\to 0^+} x\ln x = 0$,

$$\int_0^1 \ln x\,dx = \lim_{a\to 0^+}\int_a^1 \ln x\,dx = \lim_{a\to 0^+}\big(x\ln x - x\big)\Big|_{x=a}^{x=1} = \lim_{a\to 0^+}(a-1-a\ln a) = -1\,;$$

thus $\displaystyle\int_0^1 \ln x\,dx$ converges. Therefore, $\displaystyle\int_0^1 g(x)\,dx = -\ln\frac{2}{\pi} - \int_0^1 \ln x\,dx$ converges. By the direct comparison test, $\displaystyle\int_0^1 \ln\sin x\,dx$ converges.

2. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \ln\sin x\,dx$. Then the substitution of variables $u = \dfrac{\pi}{2} - x$ and $u = \pi - x$ show that

$$I = \int_0^{\frac{\pi}{2}} \ln\cos x\,dx = \int_{\frac{\pi}{2}}^{\pi} \ln\sin x\,dx\,.$$

Therefore,

$$\begin{aligned}2I &= \int_0^{\frac{\pi}{2}} \ln\sin x\,dx + \int_0^{\frac{\pi}{2}} \ln\cos x\,dx = \int_0^{\frac{\pi}{2}} \ln(\sin x\cos x)\,dx\\ &= \int_0^{\frac{\pi}{2}} \ln\sin(2x)\,dx - \int_0^{\frac{\pi}{2}} \ln 2\,dx = \int_0^{\frac{\pi}{2}} \ln\sin(2x)\,dx - \frac{\pi}{2}\ln 2\,.\end{aligned}$$

Letting $u = 2x$ so that $du = 2dx$, we obtain

$$\int_0^{\frac{\pi}{2}} \ln\sin(2x)\,dx = \frac{1}{2}\int_0^{\pi} \ln\sin u\,du = \frac{1}{2}\Big(\int_0^{\frac{\pi}{2}} \ln\sin u\,du + \int_{\frac{\pi}{2}}^{\pi} \ln\sin u\,du\Big) = I\,,$$

so $I = -\dfrac{\pi}{2}\ln 2$. □

Problem 5. Compute $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$.

Hint: Let $I(t) = \displaystyle\int_0^1 \frac{\ln(tx+1)}{x^2+1}\,dx$. Use the fact that $\displaystyle\frac{d}{dt}\int_0^1 \frac{\ln(tx+1)}{x^2+1}\,dx = \int_0^1 \frac{\partial}{\partial t}\frac{\ln(tx+1)}{x^2+1}\,dx$, where $\dfrac{\partial}{\partial t}f(x,t)$ is the derivative of f w.r.t. t variable by treating x as a constant.

Problem 6. Compute $\displaystyle\int_0^1 \frac{x-1}{\ln x}\,dx$.

Hint: Let $I(t) = \displaystyle\int_0^1 \frac{x^t-1}{\ln x}\,dx$. Use the fact that $\displaystyle\frac{d}{dt}I(t) = \int_0^1 \frac{\partial}{\partial t}\frac{x^t-1}{\ln x}\,dx$.

Problem 7. Compute $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

Hint: Let $I(t) = \displaystyle\int_0^\infty \frac{e^{-tx}\sin x}{x}\,dx$. Use the fact that $I'(t) = \displaystyle\int_0^\infty \frac{\partial}{\partial t}\frac{e^{-tx}\sin x}{x}\,dx$ and use the fact that $\displaystyle\lim_{t\to\infty} I(t) = 0$.