Exercise Problem Sets 13

Dec. 15. 2023

Problem 1. Find $\frac{d}{dx}\arcsin(\sin x)$, $\frac{d}{dx}\arccos(\sin x)$ and $\frac{d}{dx}\arctan(\tan x)$.

Problem 2. Show that $2\arcsin x = \arccos(1-2x^2)$ for all $0 \leqslant x \leqslant 1$.

Proof. **Method 1**: Let $x \in [0,1]$ be given. Then $u \equiv \arcsin x \in [0, \pi/2]$ and $\sin u = x$. Therefore, $2u \in [0,\pi] = \mathrm{Dom}(\mathrm{Cos})$; thus the double angle formula implies that

$$\mathrm{Cos}(2u) = \cos(2u) = 1 - 2\sin^2 u = 1 - 2x^2\,.$$

As a consequence,

$$2\arcsin x = 2u = \arccos(\mathrm{Cos}(2u)) = \arccos(1-2x^2)\,.$$

Method 2: Let $f(x) = 2\arcsin x - \arccos(1-2x^2)$. Then

$$\begin{aligned} f'(x) &= \frac{2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(1-2x^2)^2}}\cdot(-4x) = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{\sqrt{2x^2(2-2x^2)}} \\ &= \frac{2}{\sqrt{1-x^2}} - \frac{4x}{2x\sqrt{1-x^2}} = 0\,. \end{aligned}$$

Therefore, $f(x) = C$ for some constant C. Since $f(0) = 2\arcsin 0 - \arccos 1 = 0$, we conclude that $C = 0$ so we have $2\arcsin x - \arccos(1-2x^2)$ for all $0 \leqslant x \leqslant 1$. ▫

Problem 3. Prove the identity $\arcsin\frac{x-1}{x+1} = 2\arctan\sqrt{x} - \frac{\pi}{2}$ for all $x \geqslant 0$.

Proof. **Method 1**: Let $u = \arcsin\frac{x-1}{x+1} + \frac{\pi}{2}$. Then $u \in [0,\pi)$ and

$$\cos u = -\sin\left(u - \frac{\pi}{2}\right) = -\frac{x-1}{x+1} = \frac{2}{x+1} - 1$$

which shows that

$$x = \frac{2}{1+\cos u} - 1 = \frac{1-\cos u}{1+\cos u} = \frac{\sin^2 u/2}{\cos^2 u/2} = \tan^2\frac{u}{2}\,.$$

Therefore, by the fact that $u \in [0,\pi)$, we find that $\mathrm{Tan}\frac{u}{2} = \tan\frac{u}{2}\sqrt{x}$ so we conclude that $\frac{u}{2} = \arctan\sqrt{x}$. The desired identity follows from rearranging terms.

Method 2: Let $f(x) = \arcsin\frac{x-1}{x+1} - 2\arctan\sqrt{x}$. Then

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(x-1)^2/(x+1)^2}}\cdot\frac{d}{dx}\frac{x-1}{x+1} - 2\frac{1}{1+\sqrt{x}^2}\cdot\frac{d}{dx}\sqrt{x} \\ &= \frac{\sqrt{(x+1)^2}}{\sqrt{(x+1)^2-(x-1)^2}}\cdot\frac{(x+1)-(x-1)}{(x+1)^2} - \frac{1}{1+x}\cdot\frac{1}{\sqrt{x}} \\ &= \frac{2(x+1)}{(x+1)^2\sqrt{4x}} - \frac{1}{\sqrt{x}(1+x)} = 0\,. \end{aligned}$$

Therefore, $f(x) = C$ for some constant C. Since $f(0) = \arcsin(-1) - 2\arctan 0 = -\pi/2$, we conclude that $f(x) = -\pi/2$ which gives the desire identity. ▫

Problem 4. Prove that $\dfrac{x}{1+x^2} < \arctan x < x$ for all $x > 0$.

Proof. Let $f(x) = x - \arctan x$ and $g(x) = \arctan x - \dfrac{x}{1+x^2}$. Then for $x > 0$,

$$\begin{aligned} f'(x) &= 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0\,, \\ g'(x) &= \frac{1}{1+x^2} - \frac{(1+x^2) - 2x\cdot x}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0\,. \end{aligned}$$

Therefore, f and g are strictly increasing on $(0,\infty)$. Since f and g are continuous on $[0,\infty)$, we find that

$$f(x) > f(0) \quad \text{and} \quad g(x) > g(0) \qquad \forall\, x > 0\,.$$

The inequalities above shows the desired inequality since $f(0) = g(0) = 0$. □

Problem 5. Evaluate $\displaystyle\int_0^1 \arcsin x\, dx$ by interpreting it as an area and integrating with respect to y instead of x.

Problem 6. Evaluate the following definite integrals.

1. $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{\arcsin x}{\sqrt{1-x^2}}\, dx.$ 2. $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{\arccos x}{\sqrt{1-x^2}}\, dx.$ 3. $\displaystyle\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\, dx.$

4. $\displaystyle\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\, dx.$ 5. $\displaystyle\int_3^4 \frac{dx}{(x-1)\sqrt{x^2-2x}}.$

Solution. 5. Let $\sec u = x - 1$. Then $\sec u \tan u du = dx$; thus

$$\begin{aligned} \int_3^4 \frac{dx}{(x-1)\sqrt{x^2-2x}} &= \int_3^4 \frac{dx}{(x-1)\sqrt{(x-1)^2-1}} = \int_{\arccos\frac{1}{2}}^{\arccos\frac{1}{3}} \frac{\sec u \tan u du}{\sec u\sqrt{\sec^2 u - 1}} \\ &= \int_{\arccos\frac{1}{2}}^{\arccos\frac{1}{3}} \frac{\sec u \tan u du}{\sec u \tan u} = \int_{\arccos\frac{1}{2}}^{\arccos\frac{1}{3}} du = \arccos\frac{1}{3} - \arccos\frac{1}{2}\,. \end{aligned}$$ □

Problem 7. Find the following indefinite integrals.

1. $\displaystyle\int \sqrt{e^x - 3}\, dx.$ 2. $\displaystyle\int \frac{\sqrt{x-2}}{x+1}\, dx.$ 3. $\displaystyle\int \frac{dx}{\sqrt{-2x^2+8x+4}}.$

4. $\displaystyle\int \frac{2x\arctan(x^2+1)}{x^4+2x^2+2}\, dx.$ 5. $\displaystyle\int \frac{\sqrt{x}}{4+x^3}\, dx.$ 6. $\displaystyle\int \sqrt{\frac{x}{4+x^3}}\, dx,\ x > 0.$

Solution. 4. Let $u = \arctan(x^2+1)$. Then

$$du = \frac{1}{1+(x^2+1)^2}\frac{d}{dx}(x^2+1)\, dx = \frac{2x}{x^4+2x^2+2}\, dx\,;$$

thus

$$\int \frac{2x\arctan(x^2+1)}{x^4+2x^2+2}\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\big[\arctan(x^2+1)\big]^2 + C\,.$$ □

Problem 8. Find the function y satisfying $(1+x^2)y' + xy = 1$ and $y(0) = 1$.