Exercise Problem Sets 12

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Problem 1. Evaluate the following limits. Use L'Hôpital's Rule where appropriate. If L'Hôpital's Rule does not apply, explain why.

1. $\lim\limits_{x\to 0^+} \dfrac{\arctan(2x)}{\ln x}$. 2. $\lim\limits_{x\to 0^+} \dfrac{x^x - 1}{\ln x + x - 1}$. 3. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{\cos x + e^x - 1}$.

4. $\lim\limits_{x\to 0} \dfrac{x^a - 1}{x^b - 1}$, where $b \neq 0$. 5. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$. 6. $\lim\limits_{x\to a^+} \dfrac{\cos x \cdot \ln(x-a)}{\ln(e^x - e^a)}$.

7. $\lim\limits_{x\to 0^+} \Big(\dfrac{1}{x} - \dfrac{1}{\arctan x}\Big)$. 8. $\lim\limits_{x\to\infty} (x - \ln x)$. 9. $\lim\limits_{x\to 1^+} \ln(x^7 - 1) - \ln(x^5 - 1)$.

10. $\lim\limits_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}}$. 11. $\lim\limits_{x\to\infty} x^{e^{-x}}$. 12. $\lim\limits_{x\to 1} (2-x)^{\tan(\pi x/2)}$. 13. $\lim\limits_{x\to 0^+} (\sin x)(\ln x)$.

Problem 2. Evaluate the following limits:

1. $\lim\limits_{x\to\infty} x\Big[\big(1+\dfrac{1}{x}\big)^x - e\Big]$. 2. $\lim\limits_{x\to\infty} \Big\{\dfrac{e}{2}x + x^2\Big[\big(1+\dfrac{1}{x}\big)^x - e\Big]\Big\}$.

3. $\lim\limits_{x\to\infty} x\Big[\big(1+\dfrac{1}{x}\big)^x - e\ln\big(1+\dfrac{1}{x}\big)^x\Big]$. 4. $\lim\limits_{x\to 0} \big(\dfrac{\sin x}{x}\big)^{\frac{1}{x^2}}$. 5. $\lim\limits_{x\to\infty} \big(\dfrac{\sin x}{x}\big)^{\frac{1}{x^2}}$.

6. $\lim\limits_{x\to\infty} \Big(x - x^2 \ln\dfrac{1+x}{x}\Big)$. 7. $\lim\limits_{x\to\infty} \Big[\dfrac{1}{x}\cdot\dfrac{a^x - 1}{a-1}\Big]^{\frac{1}{x}}$, where $a > 0$ and $a \neq 1$.

Solution. 1. Let $f(x) = \big(1+\frac{1}{x}\big)^x - e$ and $g(x) = \frac{1}{x}$. Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$ and

$$f'(x) = \big(1+\frac{1}{x}\big)^x\Big[\ln\big(1+\frac{1}{x}\big) - \frac{1}{1+x}\Big] \qquad \text{and} \qquad g'(x) = -\frac{1}{x^2}\,.$$

Since

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\Big[\ln\big(1+\frac{1}{x}\big) - \frac{1}{1+x}\Big]}{\frac{d}{dx}\big(-\frac{1}{x^2}\big)} = \lim_{x\to\infty} \frac{-\frac{1}{x(1+x)^2}}{\frac{2}{x^3}} = -\frac{1}{2}\,,$$

by the fact that $\lim\limits_{x\to\infty} \ln\big(1+\frac{1}{x}\big) - \frac{1}{1+x} = \lim\limits_{x\to\infty} \frac{1}{x^2} = 0$, L'Hôpital's rule implies that

$$\lim_{x\to\infty} \frac{\Big[\ln\big(1+\frac{1}{x}\big) - \frac{1}{1+x}\Big]}{\big(-\frac{1}{x^2}\big)} = \lim_{x\to\infty} \frac{\frac{d}{dx}\Big[\ln\big(1+\frac{1}{x}\big) - \frac{1}{1+x}\Big]}{\frac{d}{dx}\big(-\frac{1}{x^2}\big)} = -\frac{1}{2}\,. \tag{$\star$}$$

Therefore,

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \big(1+\frac{1}{x}\big)^x \lim_{x\to\infty} \frac{\Big[\ln\big(1+\frac{1}{x}\big) - \frac{1}{1+x}\Big]}{\big(-\frac{1}{x^2}\big)} = -\frac{e}{2}$$

which, by L'Hôpital's rule again, implies that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = -\frac{e}{2}\,.$$

6. Let $f(x) = 1 - x\ln\frac{1+x}{x}$ and $g(x) = \frac{1}{x}$. Then $x - x^2\ln\frac{1+x}{x} = \frac{f(x)}{g(x)}$ for all $x > 0$. It is clear that $\lim_{x\to\infty} g(x) = 0$, and the fact that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\frac{1}{x}} = e$ implies that

$$\lim_{x\to\infty} f(x) = 0\,.$$

Now we compute f' and g' and obtain that

$$f'(x) = -\ln\frac{1+x}{x} - x\frac{d}{dx}\Big[\ln(1+x) - \ln x\Big] = \frac{1}{1+x} - \ln\frac{1+x}{x}$$

and $g'(x) = -\frac{1}{x^2}$. $(\star)$ then implies that $\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \frac{1}{2}$; thus L'Hôpital's rule shows that

$$\lim_{x\to\infty}\left(x - x^2\ln\frac{1+x}{x}\right) = \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \frac{1}{2}\,.$$

7. First we rewrite the function $\left[\frac{1}{x}\frac{a^x-1}{a-1}\right]^{\frac{1}{x}}$ as $\exp\left(\frac{1}{x}\ln\left(\frac{1}{x}\frac{a^x-1}{a-1}\right)\right)$ and by the continuity of exp we first compute

$$\lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{1}{x}\cdot\frac{a^x-1}{a-1}\right).$$

Nevertheless, for $x > 0$ we have $\ln\left(\frac{1}{x}\cdot\frac{a^x-1}{a-1}\right) = -\ln x + \ln|a^x-1| - \ln|a-1|$. Therefore, by the fact that

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\ln|a-1|}{x} = 0\,,$$

it suffices to compute $\lim_{x\to\infty}\frac{\ln|a^x-1|}{x}$.

The case $a > 1$: In this case, we have $\lim_{x\to\infty}\ln|a^x-1| = \lim_{x\to\infty} x = \infty$. Moreover,

$$\lim_{x\to\infty}\frac{\frac{d}{dx}\ln|a^x-1|}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{a^x\ln a}{a^x-1} = \lim_{x\to\infty}\left(\ln a + \frac{\ln a}{a^x-1}\right) = \ln a\,.$$

Therefore, L'Hôpital's rule shows that $\lim_{x\to\infty}\frac{\ln|a^x-1|}{x} = \ln a$ which further implies that

$$\lim_{x\to\infty}\left[\frac{1}{x}\cdot\frac{a^x-1}{a-1}\right]^{\frac{1}{x}} = \exp(\ln a) = a\,.$$

The case $0 < a < 1$: In this case we have $\lim_{x\to\infty}\ln|a^x-1| = 0$ and $\lim_{x\to\infty} x = \infty$; thus

$$\lim_{x\to\infty}\frac{\ln|a^x-1|}{x} = 0\,.$$

Therefore, $\lim_{x\to\infty}\left[\frac{1}{x}\cdot\frac{a^x-1}{a-1}\right]^{\frac{1}{x}} = \exp(0) = 1$.

As a consequence,

$$\lim_{x\to\infty}\left[\frac{1}{x}\cdot\frac{a^x-1}{a-1}\right]^{\frac{1}{x}} = \begin{cases} a & \text{if } a > 1\,, \\ 1 & \text{if } 0 < a < 1. \end{cases}$$

□

Problem 3. In L'Hôpital's 1696 calculus textbook, he illustrated his rule using the limit of the function

$$f(x) = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, $a > 0$. Find this limit.

Problem 4. For what values of a and b is the following equations true?

1. $\lim\limits_{x\to 0}\Big(\dfrac{\sin 2x}{x^3} + a + \dfrac{b}{x^2}\Big) = 0.$

2. $\lim\limits_{x\to 0}\Big(\dfrac{\tan 2x}{x^3} + \dfrac{a}{x^2} + \dfrac{\sin bx}{x}\Big) = 0.$

Proof. 2. Using the limit $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$, we find that $\lim\limits_{x\to 0}\dfrac{\sin bx}{x} = b$. Therefore,

$$\lim_{x\to 0}\Big(\frac{\tan 2x}{x^3} + \frac{a}{x^2}\Big) = -b\,. \tag{$\star\star$}$$

Since $\dfrac{\tan 2x}{x^3} + \dfrac{a}{x^2} = \dfrac{\tan 2x + ax}{x^3}$, if $a = -2$, then the facts that $\lim\limits_{x\to 0}(\tan 2x - 2x) = \lim\limits_{x\to 0} x^3 = 0$ and

$$\lim_{x\to 0}\frac{\frac{d}{dx}(\tan 2x - 2x)}{\frac{d}{dx}x^3} = \lim_{x\to 0}\frac{2\sec^2 2x - 2}{3x^2} = \frac{2}{3}\lim_{x\to 0}\frac{1-\cos^2 2x}{x^2\cos^2 2x} = \frac{8}{3}\lim_{x\to 0}\frac{1+\cos 2x}{\cos^2 2x}\cdot\frac{1-\cos 2x}{4x^2}$$
$$= \frac{8}{3}\,,$$

where we have used the limit $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$ to conclude that last equality. L'Hôpital's Rule then implies that

$$\lim_{x\to 0}\Big(\frac{\tan 2x}{x^3} + \frac{-2}{x^2}\Big) = \frac{8}{3}\,.$$

On the other hand, if $a \neq -2$, by the computation above we have

$$\lim_{x\to 0}\Big(\frac{\tan 2x}{x^3} + \frac{a}{x^2}\Big) = \lim_{x\to 0}\Big(\frac{\tan 2x}{x^3} + \frac{-2}{x^2} + \frac{a+2}{x^2}\Big) = \frac{8}{3} + \lim_{x\to 0}\frac{a+2}{x^2}$$

which does not exists, a contradiction to ($\star\star$). Therefore, $a = -2$ and $b = -\dfrac{8}{3}$. □

Problem 5. Show that $\lim\limits_{x\to\infty} x^{x^{-n}} = 1$ for every positive integer n.

Problem 6. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$

1. Find $f'(0)$. Is f continuously differentiable?

2. Show that f has derivatives of all orders on $\mathbb{R}$; that is, f is infinitely many times differentiable on $\mathbb{R}$.

Hint: First show by induction that there is a polynomial $p_n(x)$ and a non-negative integer k_n such that $f^{(n)}(x) = \dfrac{p_n(x)f(x)}{x^{k_n}}$ for $x \neq 0$.

Proof. 1. By the definition of the derivative,

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x}\,.$$

Since $\lim_{y\to\pm\infty} e^{y^2} = \infty$, $\lim_{y\to\pm\infty} y = \pm\infty$ and

$$\lim_{y\to\pm\infty} \frac{\frac{d}{dy}y}{\frac{d}{dy}e^{y^2}} = \lim_{y\to\pm\infty} \frac{1}{2ye^{y^2}} = 0\,, \tag{$\star\star\star$}$$

L'Hôpital's Rule shows that

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x} = \lim_{y\to\infty} \frac{e^{-y^2}}{1/y} = \lim_{y\to\infty} \frac{y}{e^{y^2}} = 0 \quad \text{and} \quad \lim_{x\to 0^-} \frac{e^{-1/x^2}}{x} = \lim_{y\to-\infty} \frac{e^{-y^2}}{1/y} = \lim_{y\to-\infty} \frac{y}{e^{y^2}} = 0\,.$$

Therefore, $\lim_{x\to 0} \frac{e^{-1/x^2}}{x} = 0$ so that $f'(0) = 0$.
On the other hand, if $x \neq 0$, we can apply the chain rule to find that

$$f'(x) = \frac{d}{dx} \exp\Big(-\frac{1}{x^2}\Big) = \exp\Big(-\frac{1}{x^2}\Big)\frac{d}{dx}\Big(-\frac{1}{x^2}\Big) = \frac{2}{x^3}e^{-1/x^2}\,.$$

Therefore,

$$f'(x) = \begin{cases} \dfrac{2}{x^3}e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

To see if f is continuously differentiable, it suffices to check if $\lim_{x\to 0} f'(x) = f'(0)$ since it is obviously that f' is continuous on $\mathbb{R}\backslash\{0\}$. Nevertheless, by the facts that $\lim_{y\to\pm\infty} y^3 = \pm\infty$, $\lim_{y\to\pm\infty} e^{y^2} = \infty$ and

$$\lim_{y\to\pm\infty} \frac{\frac{d}{dy}y^3}{\frac{d}{dy}e^{y^2}} = \lim_{y\to\pm\infty} \frac{3y^2}{2ye^{y^2}} = \frac{3}{2}\lim_{y\to\pm\infty} \frac{y}{e^{y^2}} = 0\,,$$

where the last equality is concluded from ($\star\star\star$), L'Hôpital's Rule implies that

$$\lim_{x\to 0^+} \frac{2e^{-1/x^2}}{x^3} = \lim_{y\to\infty} \frac{2y^3}{e^{y^2}} = 0 \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{e^{-1/x^2}}{x} = \lim_{y\to-\infty} \frac{2y^3}{e^{y^2}} == 0\,,$$

Therefore, $\lim_{x\to 0} f'(x) = 0 = f'(0)$ which shows that f is continuously differentiable.

2. Clearly f is infinitely many times differentiable on $\mathbb{R}\backslash\{0\}$. It suffices to show that $f^{(k)}(0)$ exists for all $k \in \mathbb{N}$. In fact, in the following we prove that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. We prove by claiming that

> for each $n \in \mathbb{N}$ there exist a polynomial p_n satisfying $p_n(0) \neq 0$ and a natural number k_n such that $f^{(n)}(x) = \dfrac{p_n(x)f(x)}{x^{k_n}}$ for $x \neq 0$.

Nevertheless, from part 1 we have $p_1(x) = 2$ and $k_1 = 2$. Now suppose that $f^{(n)}(x) = \dfrac{p_n(x)f(x)}{x^{k_n}}$ for $x \neq 0$ for some polynomial p_n satisfying $p_n(0) \neq 0$ and natural number k_n. Then for $x \neq 0$,

$$\begin{aligned} f^{(n+1)}(x) &= \frac{d}{dx}\frac{p_n(x)e^{-1/x^2}}{x^{k_n}} = \frac{\left[p_n'(x)e^{-1/x^2} + p_n(x)\cdot\frac{2}{x^3}e^{-1/x^2}\right]x^{k_n} - k_n x^{k_n-1}p_n(x)e^{-1/x^2}}{x^{2k_n}} \\ &= \frac{\left[x^3 p_n'(x) + 2p_n(x) - k_n x^2 p_n(x)\right]e^{-1/x^2}}{x^{k_n+3}} \equiv \frac{p_{n+1}(x)e^{-1/x^2}}{x^{k_n+3}}\,, \end{aligned}$$

where $p_{n+1}(x) = x^3p_n'(x) + 2p_n(x) - k_nx^2p_n(x)$ is a polynomial satisfying $p_{n+1}(0) = 2p_n(0) \neq 0$, and $k_{n+1} = k_n + 3$.

Next we claim that

$$\lim_{y\to\pm\infty}\frac{y^{m-1}}{e^y} = 0 \tag{$\diamond$}$$

for all $m \in \mathbb{N}$. Clearly ($\diamond$) holds for $m = 1$. Suppose that ($\diamond$) holds for $m = n$. Then for $m = n + 1$, by the fact that

$$\lim_{y\to\pm\infty}\frac{\frac{d}{dy}y^n}{\frac{d}{dy}e^y} = \lim_{y\to\pm\infty}\frac{ny^{n-1}}{e^y} = 0\,,$$

where the last equality follows from the induction hypothesis. Therefore, L'Hôpital's Rule implies that

$$\lim_{y\to\pm\infty}\frac{y^n}{e^y} = 0$$

which shows that ($\diamond$) holds for $m = n + 1$. By induction, we find that ($\diamond$) holds for all $m \in \mathbb{N}$.

Finally we prove $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$ by induction. From part 1 we find that $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

$$f^{(n+1)}(0) = \lim_{x\to 0}\frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0}\frac{p_n(x)f(x)}{x^{k_n+1}}\,.$$

Similar to part 1, we consider two one-sided limits $\lim\limits_{x\to 0^+}\dfrac{p_n(x)f(x)}{x^{k_n+1}}$ and $\lim\limits_{x\to 0^-}\dfrac{p_n(x)f(x)}{x^{k_n+1}}$. Suppose that $p_n(x) = a_mx^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$, where $a_0 \neq 0$. Then

$$\lim_{x\to 0^+}\frac{f(x)}{x^{k_n+1}} = \lim_{y\to\infty}\frac{y^{k_n+1}}{e^{y^2}} = \lim_{y\to\infty}\frac{1}{e^{y^2-y}}\lim_{y\to\infty}\frac{y^{k_n+1}}{e^y} = 0$$

so that

$$\lim_{x\to 0^+}\frac{p_n(x)f(x)}{x^{k_n+1}} = p_n(0)\lim_{x\to 0^+}\frac{f(x)}{x^{k_n+1}} = 0\,.$$

Similarly, $\lim\limits_{x\to 0^-}\dfrac{p_n(x)f(x)}{x^{k_n+1}} = 0$; thus $f^{(n+1)}(0) = 0$. By induction we conclude that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. □