Exercise Problem Sets 10

Dec. 1. 2023

Problem 1. Find the following integrals.

1. $\displaystyle\int \frac{x-1}{x^2-4x-5}\,dx$ 2. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx$ 3. $\displaystyle\int \frac{1}{x\sqrt{4x+1}}\,dx$ 4. $\displaystyle\int \frac{1}{x+4+4\sqrt{x+1}}\,dx$

5. $\displaystyle\int \frac{1}{3-2\sin x}\,dx$ 6. $\displaystyle\int \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$ 7. $\displaystyle\int \frac{4}{\tan x-\sec x}\,dx$ 8. $\displaystyle\int \frac{1+4\cot x}{4-\cot x}\,dx$

Solution. 7. Note that the integrand is identical to $\dfrac{4\cos x}{\sin x-1}$.

$$\begin{aligned}\int \frac{4}{\tan x-\sec x}\,dx &= \int \frac{4\cos x}{\sin x-1}\,dx = \int \frac{4}{u}\,du = 4\ln|u|+C = 4\ln|\sin x-1|+C\\ &= 4\ln(1-\sin x)+C\,.\end{aligned}$$

□

Problem 2. Show that the following functions are decreasing on $(0,\infty)$.

1. $y=\big(1+\frac{1}{2x}\big)^{x+0.5}$. 2. $y=\big(1+\frac{1}{x}\big)^{x+0.5}$.

Proof. 1. Let $f(x)=\big(1+\frac{1}{2x}\big)^{x+0.5}$. Then

$$\begin{aligned}f'(x) &= \frac{d}{dx}\exp\Big((x+0.5)\ln\big(1+\frac{1}{2x}\big)\Big)\\ &= \exp\Big((x+0.5)\ln\big(1+\frac{1}{2x}\big)\Big)\Big[\ln\big(1+\frac{1}{2x}\big)+(x+0.5)\frac{1}{1+\frac{1}{x}}\frac{-1}{2x^2}\Big]\\ &= f(x)\Big[\ln\big(1+\frac{1}{2x}\big)-\frac{1}{2(x+1)}-\frac{1}{4}\frac{1}{x^2+x}\Big]\,.\end{aligned}$$

Let $g(x)=\ln\big(1+\frac{1}{2x}\big)-\dfrac{1}{2(x+1)}-\dfrac{1}{4}\dfrac{1}{x^2+x}$. Then

$$\begin{aligned}g'(x) &= \frac{1}{1+\frac{1}{2x}}\frac{-1}{2x^2}+\frac{1}{2(1+x)^2}+\frac{1}{4}\frac{2x+1}{(x^2+x)^2} = \frac{-1}{2x(x+1)}+\frac{1}{2(1+x)^2}+\frac{1}{4}\frac{2x+1}{x^2(x+1)^2}\\ &= \frac{-2x(x+1)+2x^2+2x+1}{4x^2(x+1)^2} = \frac{1}{4x^2(x+1)^2} > 0\,.\end{aligned}$$

Therefore, g is strictly increasing on $(0,\infty)$; thus

$$g(x)\leqslant \lim_{x\to\infty} g(x) = \lim_{x\to\infty}\Big[\ln\big(1+\frac{1}{2x}\big)-\frac{1}{2(x+1)}-\frac{1}{4}\frac{1}{x^2+x}\Big] = \ln 1 = 0\,.$$

Since $f(x)>0$ for all $x>0$, we conclude that $f'(x)\leqslant 0$ for all $x>0$; thus f is decreasing on $(0,\infty)$.

2. Let $f(x) = \left(1+\frac{1}{x}\right)^{x+0.5}$. Then

$$\begin{aligned} f'(x) &= \frac{d}{dx}\exp\Big((x+0.5)\ln\big(1+\frac{1}{x}\big)\Big) \\ &= \exp\Big((x+0.5)\ln\big(1+\frac{1}{x}\big)\Big)\Big[\ln\big(1+\frac{1}{x}\big)+(x+0.5)\frac{1}{1+\frac{1}{x}}\frac{-1}{x^2}\Big] \\ &= f(x)\Big[\ln\big(1+\frac{1}{x}\big)-\frac{1}{x+1}-\frac{1}{2}\frac{1}{x^2+x}\Big]. \end{aligned}$$

Let $g(x) = \ln\big(1+\frac{1}{x}\big)-\frac{1}{x+1}-\frac{1}{2}\frac{1}{x^2+x}$. Then

$$\begin{aligned} g'(x) &= \frac{1}{1+\frac{1}{x}}\frac{-1}{x^2}+\frac{1}{(1+x)^2}+\frac{1}{2}\frac{2x+1}{(x^2+x)^2} = \frac{-1}{x(x+1)}+\frac{1}{(1+x)^2}+\frac{1}{2}\frac{2x+1}{x^2(x+1)^2} \\ &= \frac{-2x(x+1)+2x^2+2x+1}{2x^2(x+1)^2} = \frac{1}{2x^2(x+1)^2} > 0\,. \end{aligned}$$

Therefore, g is strictly increasing on $(0,\infty)$; thus

$$g(x) \leqslant \lim_{x\to\infty} g(x) = \lim_{x\to\infty}\Big[\ln\big(1+\frac{1}{x}\big)-\frac{1}{x+1}-\frac{1}{2}\frac{1}{x^2+x}\Big] = \ln 1 = 0\,.$$

Since $f(x) > 0$ for all $x > 0$, we conclude that $f'(x) \leqslant 0$ for all $x > 0$; thus f is decreasing on $(0,\infty)$.

We note that the two functions given above are indeed strictly decreasing since for $0 < x < y < z$, in both case we have

$$g(x) < g(y) < g(z)\,;$$

thus passing to the limit as $z \to \infty$ we obtain that $g(x) < g(y) \leqslant 0$ for all $0 < x < y$. This shows that $g(x) < 0$ for all $x > 0$; thus $f'(x) < 0$ for all $x > 0$ which shows that f is strictly decreasing on $(0,\infty)$. □

Problem 3. In this example you are asked to compute the integral of $y = xe^x$ by the Riemann sum. Complete the following.

1. Show that if $r \neq 1$, then $\sum\limits_{k=1}^{n} kr^k = \frac{r(1-r^n)}{(1-r)^2} - \frac{nr^{n+1}}{1-r}$.

2. Compute $\int_0^a xe^x\,dx$ by the limit the Riemann sum of $y = xe^x$ for regular partition using the right end-point rule.

3. Find an anti-derivative of $y = xe^x$.

Problem 4 (Integrating Factor).

1. Let $f, g : [a,b] \to \mathbb{R}$ be a continuous function, F be an anti-derivative of f, and $y : [a,b] \to \mathbb{R}$ satisfies that

$$y' + f(x)y = g(x)\,. \tag{$\star$}$$

Find an expression of y.

2. Find the function y satisfying $y' + x^2 y = 2x^2$ and $y(0) = 1$.

Solution. 1. Let F be an anti-derivative of f. Then $F' = f$ so that

$$\frac{d}{dx}\big[e^{F(x)}y(x)\big] = e^{F(x)}F'(x)y(x) + e^{F(x)}y'(x) = e^{F(x)}\big[y'(x) + f(x)y(x)\big] = e^{F(x)}g(x)\,.$$

Therefore, $e^F y$ is an anti-derivative of $e^F g$ which shows that

$$y(x) = e^{-F(x)} \int e^{F(x)} g(x)\,dx\,.$$

2. Let $F(x) = \dfrac{x^3}{3}$. Then $F'(x) = x^2$; thus

$$\frac{d}{dx}\big[e^{x^3/3}y(x)\big] = 2x^2$$

which shows that

$$e^{x^3/3}y(x) = \int 2x^2 e^{x^3/3}\,dx \overset{u=x^3/3}{=} \int 2e^u\,du = 2e^u + C = 2e^{x^3/3} + C\,.$$

Therefore,

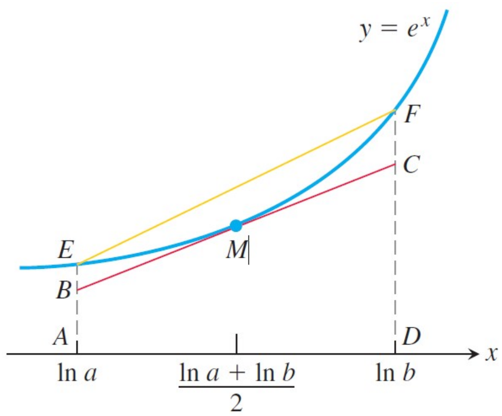
$$y(x) = 2 + Ce^{-x^3/3}\,.$$

Since $y(0) = 1$, we conclude that $C = -1$ so that $y(x) = 2 - e^{-x^3/3}$. □

Hint: Multiply both sides of $(\star)$ by $\exp(F(x))$ and observe that the left-hand side is the derivative of a certain function, where F is an anti-derivative of f.

Problem 5. 1. Show that for $0 < a < b$,

$$e^{\frac{\ln a + \ln b}{2}} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x\,dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a)\,.$$



2. Using the result above to show that for $0 < a < b$,

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}\,.$$

Problem 6. Prove the following inequalities.

1. $e^x > 1 + \ln(1+x)$ for all $x > 0$.

2. $e^x > 1 + (1+x)\ln(1+x)$ for all $x > 0$.

3. $e^x \geqslant 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$ for all $x \geqslant 0$ and $n \in \mathbb{N}$.

Problem 7. Let a, b be two positive numbers, p, q any nonzero numbers, and $p < q$. Prove that

$$\big[\theta a^p + (1-\theta)b^p\big]^{\frac{1}{p}} \leqslant \big[\theta a^q + (1-\theta)b^q\big]^{\frac{1}{q}} \qquad \forall\, \theta \in (0,1)\,.$$

Hint: Show that the function $f(p) = \big[\theta a^p + (1-\theta)b^p\big]^{\frac{1}{p}}$ is an increasing function of p.

Proof. Let $\theta \in (0,1)$ be given, and define $f(p) = \big[\theta a^p + (1-\theta)b^p\big]^{\frac{1}{p}}$. Then

$$\begin{aligned}
f'(p) &= \frac{d}{dp}\big[\theta a^p + (1-\theta)b^p\big]^{\frac{1}{p}} = \frac{d}{dp}\exp\Big(\frac{\ln\big[\theta a^p + (1-\theta)b^p\big]}{p}\Big) \\
&= \exp\Big(\frac{\ln\big[\theta a^p + (1-\theta)b^p\big]}{p}\Big)\frac{\frac{\theta a^p \ln a + (1-\theta)b^p \ln b}{\theta a^p + (1-\theta)b^p}p - \ln\big[\theta a^p + (1-\theta)b^p\big]}{p^2} \\
&= f(p)\frac{\theta a^p \ln(a^p) + (1-\theta)b^p\ln(b^p) - \big[\theta a^p + (1-\theta)b^p\big]\ln\big[\theta a^p + (1-\theta)b^p\big]}{p^2\big[\theta a^p + (1-\theta)b^p\big]} \\
&= f(p)\frac{(1-\theta)b^p\big[\ln(b^p) - \ln(a^p)\big] - \big[\theta a^p + (1-\theta)b^p\big]\ln\big[\theta + (1-\theta)(b/a)^p\big]}{p^2\big[\theta a^p + (1-\theta)b^p\big]} \\
&= \frac{a^p f(p)}{p^2\big[\theta a^p + (1-\theta)b^p\big]}\Big[(1-\theta)(b/a)^p\ln(b/a)^p - \big[\theta + (1-\theta)(b/a)^p\big]\ln\big[\theta + (1-\theta)(b/a)^p\big]\Big]
\end{aligned}$$

Let $g(x) = (1-\theta)x\ln x - \big[\theta + (1-\theta)x\big]\ln\big[\theta + (1-\theta)x\big]$. Then the above identity shows that

$$f'(p) = \frac{a^p f(p)}{p^2\big[\theta a^p + (1-\theta)b^p\big]}g\big((b/a)^p\big)\,;$$

thus to show that f is increasing on $(0,\infty)$ it suffices to shows that $g(x) \geqslant 0$ for $x > 0$. Compute the first and the second derivative of g, we find that

$$\begin{aligned}
g'(x) &= (1-\theta)(\ln x + 1) - (1-\theta)\ln\big[\theta + (1-\theta)x\big] - \frac{\theta + (1-\theta)x}{\theta + (1-\theta)x}\cdot(1-\theta) \\
&= (1-\theta)\ln x - (1-\theta)\ln\big[\theta + (1-\theta)x\big] = (1-\theta)\ln\frac{x}{\theta + (1-\theta)x}
\end{aligned}$$

and

$$g''(x) = (1-\theta)\Big(\frac{1}{x} - \frac{1-\theta}{\theta + (1-\theta)x}\Big) = \frac{\theta(1-\theta)}{x\big[\theta + (1-\theta)x\big]}\,.$$

Therefore, $g'(x) = 0$ if and only if $\dfrac{x}{\theta + (1-\theta)x} = 1$ or equivalently, $x = 1$. This shows that $x = 1$ is the only critical point of g, and the second derivative test implies that g attains its absolute minimum at $x = 1$. Therefore, for $x > 0$ we have

$$g(x) \geqslant g(1) = (1-\theta)\ln 1 - \big[\theta + (1-\theta)\big]\ln\big[\theta + (1-\theta)\big] = 0$$

which concludes that f is increasing on $(0,\infty)$. □

Problem 8. 1. Find an equation for the line through the origin tangent to the graph of $y = \ln x$.

2. Show that $\ln x < \dfrac{x}{e}$ for all $x \neq e$.

3. Show that $x^e < e^x$ for all $x \neq e$.

4. Show that if $e \leqslant A < B$, then $A^B > B^A$.

Proof. 1. Suppose that the tangent point is $(a, \ln a)$. Then the tangent line of the graph of $y = \ln x$ passing through $(a, \ln a)$ is given by

$$y = \frac{1}{a}(x - a) + \ln a\,.$$

Since the tangent line passing through $(0, 0)$, we must have $0 = -1 + \ln a$ which shows that $a = e$. Therefore, the tangent line of the graph of $y = \ln x$ passing through $(0, 0)$ is given by

$$y = \frac{1}{e}(x - e) + \ln e = \frac{x}{e}\,.$$

2. Graphically speaking, since the graph of $y = \ln x$ is (strictly) concave downward, any tangent line of the graph of $y = \ln x$ lies above the graph of $y = \ln x$; thus we have $\ln x \leqslant \frac{x}{e}$ for all $x > 0$, and the strictly concavity of the graph of f shows that $\ln x < \dfrac{x}{e}$ for all $x > 0$ and $x \neq e$.

To see this analytically, let $f(x) = \ln x - \dfrac{x}{e}$. Then $f'(x) = \dfrac{1}{x} - \dfrac{1}{e}$; thus $x = e$ is the only critical point of f and

$$f'(x) > 0 \quad \text{if } 0 < x < e \qquad \text{and} \qquad f'(x) < 0 \quad \text{if } x > e\,.$$

The first derivative test shows that f attains its absolute minimum at e. Moreover, f is strictly increasing on $(0, e)$ and is strictly decreasing on (e, ∞). Therefore,

$$f(x) < f(e) = 0 \quad \text{on } (0, e) \qquad \text{and} \qquad f(x) < f(e) = 0 \quad \text{on } (e, \infty)$$

so we conclude that $\ln x < \dfrac{x}{e}$ for all $x > 0$ and $x \neq e$.

3. Note that part 2 shows that

$$\ln(x^e) = e \ln x < x = x \ln e = \ln(e^x) \qquad \forall\, x > 0 \text{ and } x \neq e\,.$$

Therefore, since ln is strictly increasing on $(0, \infty)$, we have

$$x^e < e^x \qquad \forall\, x > 0 \text{ and } x \neq e\,.$$

4. Here we present two ways of proving part 4.

(a) From part 2 and the logarithmic properties of ln, we have

$$\ln e + \ln x \ln(ex) < \frac{ex}{e} = x \qquad \forall\, x > 0 \text{ and } x \neq 1\,.$$

Therefore,

$$\ln x < x - 1 \qquad \forall\, x > 0 \text{ and } x \neq 1\,.$$

Let $e \geqslant A < B$. Then $\frac{B}{A} > 1$; thus the inequality above shows that

$$\ln\frac{B}{A} < \frac{B}{A} - 1 < \Big(\frac{B}{A} - 1\Big)\ln A\,,$$

where the last inequality follows from the fact that $\ln A > 1$ and $\frac{B}{A} - 1 > 0$. Therefore,

$$\ln B - \ln A < \frac{B}{A}\ln A - \ln A$$

which shows that $A \ln B < B \ln A$. Therefore,

$$\ln(B^A) = A\ln B < B\ln A = \ln(A^B)$$

and the fact that ln is strictly increasing implies that $B^A < A^B$.

(b) Let $A \geqslant e$ be given, and let $f(x) = A\ln x - x\ln A$ on $[A, \infty)$. Then

$$f'(x) = \frac{A}{x} - \ln A = \frac{\ln A}{x}\Big(\frac{A}{\ln A} - x\Big)\,.$$

Since $A \geqslant e$, we have $\frac{A}{\ln A} < A$; thus $f'(x) < 0$ if $x \in [A, \infty)$. This shows that f is strictly decreasing on $[A, \infty)$; thus if $B > A$, we have $f(A) > f(B)$ which implies that

$$0 = A\ln A - A\ln A = f(A) > f(B) = A\ln B - B\ln A$$

Therefore, $B\ln A > A\ln B$ which gives the desired result. □

Problem 9 (Implicit Differentiation).

1. Find y' if $e^{\frac{x}{y}} = x - y$.

2. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

3. Find an equation of the tangent line to the curve $1 + \ln xy = e^{x-y}$ at the point $(1, 1)$