Calculus 微積分

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Chapter 9

Infinite Series

9.1 Sequences

Definition 9.1: Sequence

A ***sequence*** of real numbers (or simply a real sequence) is a function $f : \mathbb{N} \to \mathbb{R}$. The collection of numbers $\big\{f(1), f(2), f(3), \cdots\big\}$ are called ***terms*** of the sequence and the value of f at n is called the ***n-th term*** of the sequence. We usually use f_n to denote the n-th term of a sequence $f : \mathbb{N} \to \mathbb{R}$, and this sequence is usually denoted by $\{f_n\}_{n=1}^{\infty}$ or simply $\{f_n\}$.

Example 9.2. Let $f : \mathbb{N} \to \mathbb{R}$ be the sequence defined by $f(n) = 3 + (-1)^n$. Then f is a real sequence. Its terms are $\{2, 4, 2, 4, \cdots\}$.

Example 9.3. A sequence can also be defined recursively. For example, let $\{a_n\}_{n=1}^{\infty}$ be defined by

$$a_{n+1} = \sqrt{2a_n}\,, \qquad a_1 = \sqrt{2}\,.$$

Then $a_2 = \sqrt{2\sqrt{2}}$, $a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$, and etc. The general form of a_n is given by

$$a_n = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}} = 2^{\frac{2^n-1}{2^n}}\,.$$

There are also sequences that are defined recursively but it is difficult to obtain the general form of the sequence. For example, let $\{b_n\}_{n=1}^{\infty}$ be defined by

$$b_{n+1} = \sqrt{2 + b_n}\,, \qquad b_1 = \sqrt{2}\,.$$

Then $b_2 = \sqrt{2 + \sqrt{2}}$, $b_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$, and etc.

Remark 9.4. Occasionally, it is convenient to begin a sequence with the 0-th term or even the k-th term. In such cases, we write $\{a_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=k}^{\infty}$ to denote the sequences.

Similar to the concept of the limit of functions, we would like to consider the limit of sequences; that is, we would like to know to which value the n-th term of a sequence approaches as n become larger and larger.

Definition 9.5

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to ***converge to*** L if for every $\varepsilon > 0$, there exists $N > 0$ such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n \geqslant N\,.$$

Such an L (must be a real number and) is called a ***limit*** of the sequence. If $\{a_n\}_{n=1}^{\infty}$ converges to L, we write $a_n \to x$ as $n \to \infty$.
A sequence of real number $\{a_n\}_{n=1}^{\infty}$ is said to be ***convergent*** if there exists $L \in \mathbb{R}$ such that $\{a_n\}_{n=1}^{\infty}$ converges to L. If no such L exists we say that $\{a_n\}_{n=1}^{\infty}$ ***does not converge*** or simply ***diverges***.

Motivation: Intuitively, we expect that a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ converges to a number L if "outside any open interval containing L there are only finitely many $a_n's$". The statement inside " " can be translated into the following mathematical statement:

$$\forall\,\varepsilon > 0, \#\big\{n \in \mathbb{N}\,\big|\,a_n \notin (L-\varepsilon, L+\varepsilon)\big\} < \infty\,, \tag{9.1.1}$$

where $\#A$ denotes the number of points in the set A. One can easily show that the convergence of a sequence defined by (9.1.1) is equivalent to Definition 9.5.

In the definition above, we do not exclude the possibility that there are two different limits of a convergent sequence. In fact, this is never the case because of the following

Proposition 9.6

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and $a_n \to a$ and $a_n \to b$ as $n \to \infty$, then $a = b$. (若收斂則極限唯一).

We will not prove this proposition and treat it as a fact.

• **Notation**: Since the limit of a convergent sequence is unique, we use $\lim\limits_{n\to\infty} a_n$ to denote this unique limit of a convergent sequence $\{a_n\}_{n=1}^{\infty}$.

Theorem 9.7

Let L be a real number, and $f : [1, \infty) \to \mathbb{R}$ be a function of a real variable such that $\lim\limits_{x\to\infty} f(x) = L$. If $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L\,.$$

Example 9.8. The limit of the sequence $\{e_n\}_{n=1}^{\infty}$ defined by $e_n = \left(1 + \dfrac{1}{n}\right)^n$ is e.

When a sequence $\{a_n\}_{n=1}^{\infty}$ is given by evaluating a differentiable function $f : [1, \infty) \to \mathbb{R}$ on $\mathbb{N}$, sometimes we can use L'Hôspital's rule to find the limit of the sequence.

Example 9.9. The limit of the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \dfrac{n^2}{2^n - 1}$ is

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{2^x \ln 2} = \lim_{x\to\infty} \frac{2}{2^x (\ln 2)^2} = 0\,.$$

There are cases that a sequence cannot be obtained by evaluating a function defined on $[1, \infty)$. In such cases, the limit of a sequence cannot be computed using L'Hôspital's rule and it requires more techniques to find the limit.

Example 9.10. The limit of the sequence $\{s_n\}_{n=1}^{\infty}$ defined by $s_n = \dfrac{n!}{n^{n+\frac{1}{2}} e^{-n}}$ is $\sqrt{2\pi}$; that is,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\, n^n e^{-n}} = 1\,. \tag{9.1.2}$$

Similar to Theorem 1.14, we have the following

Theorem 9.11

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$. Then

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm K$.
2. $\lim\limits_{n\to\infty} (a_n b_n) = LK$. In particular, $\lim\limits_{n\to\infty} (ca_n) = cL$ if c is a real number.
3. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{K}$ if $K \neq 0$.

Theorem 9.12: Squeeze Theorem

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n \leqslant c_n \leqslant b_n$ for all $n \geqslant N$. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$, then $\lim\limits_{n\to\infty} c_n = L$.

Theorem 9.13: Absolute Value Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Let $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequence of real numbers defined by $b_n = -|a_n|$ and $c_n = |a_n|$. Then $b_n \leqslant a_n \leqslant c_n$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} |a_n| = 0$, Theorem 9.11 implies that $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = 0$ and the Squeeze Theorem further implies that $\lim\limits_{n\to\infty} a_n = 0$. □

Definition 9.14: Monotonicity of Sequences

A sequence $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is said to be

1. (**monotone**) **increasing** if $a_{n+1} \geqslant a_n$ for all $n \in \mathbb{N}$;
2. (**monotone**) **decreasing** if $a_{n+1} \leqslant a_n$ for all $n \in \mathbb{N}$;
3. ***monotone*** if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence or a decreasing sequence.

Example 9.15. The sequence $\{s_n\}_{n=2}^{\infty}$ defined in Example 9.10 is a monotone decreasing sequence.

Definition 9.16: Boundedness of Sequences

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

1. $\{a_n\}_{n=1}^{\infty}$ is said to be ***bounded***（有界的）if there exists $M \in \mathbb{R}$ such that $|a_n| \leqslant M$ for all $n \in \mathbb{N}$.
2. $\{a_n\}_{n=1}^{\infty}$ is said to be ***bounded from above***（有上界）if there exists $B \in \mathbb{R}$, called an ***upper bound*** of the sequence, such that $a_n \leqslant B$ for all $n \in \mathbb{N}$. Such a number B is called an upper bound of the sequence.
3. $\{a_n\}_{n=1}^{\infty}$ is said to be ***bounded from below***（有下界）if there exists $A \in \mathbb{R}$, called a ***lower bound*** of the sequence, such that $A \leqslant a_n$ for all $n \in \mathbb{N}$. Such a number A is called a lower bound of the sequence.

Example 9.17. The sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = n$ is bounded from below by 0 by not bounded from above.

Proposition 9.18

A convergent sequence of real numbers is bounded（數列收斂必有界）.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit L. Then by the definition of limits of sequences, there exists $N > 0$ such that

$$a_n \in (L-1, L+1) \qquad \forall\, n \geqslant N.$$

Let $M = \max\big\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |L|+1\big\}$. Then $|a_n| \leqslant M$ for all $n \in \mathbb{N}$. ☐

Remark 9.19. A bounded sequence might not be convergent. For example, let $\{a_n\}_{n=1}^{\infty}$ be defined by $a_n = 3 + (-1)^n$. Then

$$a_1 = a_3 = a_5 = \cdots = a_{2k-1} = \cdots = 2 \quad \text{and} \quad a_2 = a_4 = a_6 = \cdots = a_{2k} = \cdots = 4\,.$$

Therefore, the only possible limits are $\{2, 4\}$; however, by the fact that

$$\#\big\{n \in \mathbb{N} \,\big|\, a_n \notin (1,3)\big\} = \#\big\{n \in \mathbb{N} \,\big|\, a_n \notin (3,5)\big\} = \infty\,,$$

we find that 2 and 4 are not the limit of $\{a_n\}_{n=1}^{\infty}$. Therefore, $\{a_n\}_{n=1}^{\infty}$ does not converge.

- **Completeness of Real Numbers**:

One important property of the real numbers is that they are ***complete***. The completeness axiom for real numbers states that "every bounded sequence of real numbers has a ***least upper bound*** and a ***greatest lower bound***"; that is, if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, then there exists an upper bound M and a lower bound m of $\{a_n\}_{n=1}^{\infty}$ such that there is no smaller upper bound nor greater lower bound of $\{a_n\}_{n=1}^{\infty}$.

Theorem 9.20: Monotone Sequence Property (MSP)

Let $\{a_n\}_{n=1}^{\infty}$ be a monotone sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ converges if and only if $\{a_n\}_{n=1}^{\infty}$ is bounded.

Proof. It suffices to show the "$\Leftarrow$" direction.

Without loss of generality, we can assume that $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded. By the completeness of real numbers, there exists a least upper bound M for the sequence $\{a_n\}_{n=1}^{\infty}$.

Let $\varepsilon > 0$ be given. Since M is the least upper bound for $\{a_n\}_{n=1}^{\infty}$, $M - \varepsilon$ is not an upper bound; thus there exists $N \in \mathbb{N}$ such that $a_N > M - \varepsilon$. Since $\{a_n\}_{n=1}^{\infty}$ is increasing, $a_n \geqslant a_N$ for all $n \geqslant N$. Therefore,

$$M - \varepsilon < a_n \leqslant M \qquad \forall\, n \geqslant N$$

which implies that

$$|a_n - M| < \varepsilon \qquad \forall\, n \geqslant N\,.$$

The statement above shows that $\{a_n\}_{n=1}^{\infty}$ converges to M. □

Remark 9.21. A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is called a ***Cauchy sequence*** if for every $\varepsilon > 0$ there exists $N > 0$ such that

$$|a_n - a_m| < \varepsilon \qquad \text{whenever} \quad n, m \geqslant N\,.$$

A convergent sequence must be a Cauchy sequence. Moreover, the completeness of real numbers is equivalent to that "every Cauchy sequence of real number converges".

9.2 Series and Convergence

An infinite series is the "sum" of an infinite sequence. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots$$

is an infinite series (or simply series). The numbers a_1, a_2, a_3, $\cdots$ are called the terms of the series. For convenience, the sum could begin the index at $n = 0$ or some other integer.

Definition 9.22

The series $\sum\limits_{k=1}^{\infty} a_k$ is said to be convergent or converge to S if the sequence of the partial sum, denoted by $\{S_n\}_{n=1}^{\infty}$ and defined by

$$S_n \equiv \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n,$$

converges to S. S_n is called the n-th partial sum of the series $\sum\limits_{k=1}^{\infty} a_k$.

When the series converges, we write $S = \sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} a_k$ is said to be convergent. If $\{S_n\}_{n=1}^{\infty}$ diverges, the series is said to be divergent or diverge. If $\lim\limits_{n\to\infty} S_n = \infty$ (or $-\infty$), the series is said to diverge to ∞ (or $-\infty$).

Example 9.23. The n-th partial sum of the series $\sum\limits_{k=1}^{\infty}\dfrac{1}{k(k+1)}$ is

$$\begin{aligned} S_n &= \sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\Big(\frac{1}{k}-\frac{1}{k+1}\Big) = \Big(1-\frac{1}{2}\Big)+\Big(\frac{1}{2}-\frac{1}{3}\Big)+\cdots+\Big(\frac{1}{n}-\frac{1}{n+1}\Big) \\ &= 1-\frac{1}{n+1}\,; \end{aligned}$$

thus the series $\sum\limits_{k=1}^{\infty}\dfrac{1}{k(k+1)}$ converges to 1, and we write $\sum\limits_{k=1}^{\infty}\dfrac{1}{k(k+1)}=1$.

Example 9.24. The n-th partial sum of the series $\sum\limits_{k=1}^{\infty}\dfrac{2}{4k^2-1}$ is

$$\begin{aligned} \sum_{k=1}^{n}\frac{2}{4k^2-1} &= \sum_{k=1}^{n}\frac{2}{(2k-1)(2k+1)} = \sum_{k=1}^{n}\Big(\frac{1}{2k-1}-\frac{1}{2k+1}\Big) \\ &= \Big(1-\frac{1}{3}\Big)+\Big(\frac{1}{3}-\frac{1}{5}\Big)+\cdots+\Big(\frac{1}{2n-1}-\frac{1}{2n+1}\Big) = 1-\frac{1}{2n+1}\,; \end{aligned}$$

thus the series $\sum\limits_{k=1}^{\infty}\dfrac{2}{4k^2-1}$ converges to 1, and we write $\sum\limits_{k=1}^{\infty}\dfrac{2}{4k^2-1}=1$.

The series in the previous two examples are series of the form

$$\sum_{k=1}^{n}(b_k-b_{k+1}) = (b_1-b_2)+(b_2-b_3)+\cdots+(b_n-b_{n+1})+\cdots\,,$$

and are called telescoping series. A telescoping series converges if and only if $\lim\limits_{n\to\infty} b_n$ converges.

Example 9.25. The series $\sum\limits_{k=1}^{\infty} r^k$, where r is a real number, is called a geometric series (with ratio r). Note that the n-th partial sum of the series is

$$S_n = \sum_{k=1}^{n} r^k = 1+r+r^2+\cdots+r^n = \begin{cases} \dfrac{1-r^{n+1}}{1-r} & \text{if } r\neq 1\,, \\ n+1 & \text{if } r=1\,. \end{cases}$$

Therefore, the geometric series converges if and only if the common ratio r satisfies $|r|<1$.

Theorem 9.26

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty}$ be convergent series, and c is a real number. Then

1. $\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$.

2. $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.

3. $\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$.

Theorem 9.27: Cauchy Criteria

A series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\varepsilon > 0$, there exists $N > 0$ such that

$$\Big|\sum_{k=n}^{n+\ell} a_k\Big| < \varepsilon \qquad \text{whenever} \quad n \geqslant N, \ell \geqslant 0\,.$$

Proof. Let S_n be the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$. Then by Remark 9.21,

$$\begin{aligned}\sum_{k=1}^{\infty} a_k \text{ converges} &\Leftrightarrow \{S_n\}_{n=1}^{\infty} \text{ is a convergent sequence}\\ &\Leftrightarrow \{S_n\}_{n=1}^{\infty} \text{ is a Cauchy sequence}\\ &\Leftrightarrow \text{for every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that}\\ &\qquad\qquad |S_n - S_m| < \varepsilon \text{ whenever } n, m \geqslant N\\ &\Leftrightarrow \text{for every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that}\\ &\qquad |a_n + a_{n+1} + \cdots + a_{n+\ell}| < \varepsilon \text{ whenever } n \geqslant N \text{ and } \ell \geqslant 0.\end{aligned}$$

□

Corollary 9.28: n-th Term Test

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Remark 9.29. It is not true that $\lim_{n\to\infty} a_n = 0$ implies the convergence of $\sum_{k=1}^{\infty} a_k$. For example, we have shown in Example 8.50 that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to ∞ while we know that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Corollary 9.30: n-th term test for divergence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. If $\lim\limits_{n\to\infty} a_n \neq 0$ or does not exist, then the series $\sum\limits_{k=1}^{\infty} a_k$ diverges.

9.3 The Integral Test and p-Series

9.3.1 The integral test

Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ is obtained by evaluating a non-negative continuous decreasing function $f : [1,\infty) \to \mathbb{R}$ on $\mathbb{N}$; that is, $f(n) = a_n$. Then

$$\int_1^{n+1} f(x)\,dx \leqslant S_n \equiv \sum_{k=1}^{n} a_k \leqslant a_1 + \int_1^n f(x)\,dx\,. \tag{9.3.1}$$

Since the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ of the series $\sum\limits_{k=1}^{\infty} a_k$ is increasing, the completeness of real numbers implies that $\{S_n\}_{n=1}^{\infty}$ converges if and only if the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$ converges.

Theorem 9.31

Let $f : [1,\infty) \to \mathbb{R}$ be a non-negative continuous decreasing function. The series $\sum\limits_{k=1}^{\infty} f(k)$ converges if and only if the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$ converges.

Example 9.32. The series $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2+1}$ converges since

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \arctan x\Big|_{x=1}^{x=b} = \lim_{b\to\infty} (\arctan b - \arctan 1) = \frac{\pi}{4}$$

and the function $f(x) = \dfrac{1}{x^2+1}$ is non-negative continuous and decreasing on $[1,\infty)$.

Example 9.33. The series $\sum\limits_{k=1}^{\infty} \dfrac{k}{k^2+1}$ diverges since

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{\ln(x^2+1)}{2}\Big|_{x=1}^{x=b} = \frac{1}{2}\lim_{b\to\infty} \big[\ln(b^2+1) - \ln 2\big] = \infty$$

and the function $f(x) = \dfrac{x}{x^2+1}$ is non-negative continuous and decreasing on $[1,\infty)$.

Example 9.34. The series $\sum\limits_{k=2}^{\infty}\frac{1}{k\ln k}$ converges since

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln x} \overset{(x=e^u)}{=} \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{e^u du}{e^u \ln e^u} = \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{du}{u} = \lim_{b\to\infty}\ln u\Big|_{u=\ln 2}^{u=\ln b}$$
$$= \lim_{b\to\infty}(\ln\ln b - \ln\ln 2) = \infty$$

and the function $f(x) = \frac{1}{x\ln x}$ is non-negative continuous and decreasing on $[2,\infty)$.

9.3.2 p-series

A series of the form

$$\sum_{k=1}^{\infty}\frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is called a p-series. The series is a function of p, and this function is usually called the ***Riemann zeta function***; that is,

$$\zeta(s) \equiv \sum_{n=1}^{\infty}\frac{1}{n^s}\,.$$

A harmonic series is the p-series with $p=1$, and a general harmonic series is of the form

$$\sum_{k=1}^{\infty}\frac{1}{ak+b}\,.$$

By Theorem 8.51 and 9.31, the p-series converges if and only if $p>1$.

Remark 9.35. It can be shown that $\sum\limits_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$. In fact, for all integer $k \geqslant 2$, the number $\sum\limits_{k=1}^{\infty}\frac{1}{n^k}$ can be computed by hand (even though it is very time consuming).

Remark 9.36. Using (9.3.1), we find that

$$\ln(n+1) \leqslant \sum_{k=1}^{n}\frac{1}{k} \leqslant 1 + \ln n \qquad \forall\, n\in\mathbb{N}\,.$$

Therefore, the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$a_n = \sum_{k=1}^{n}\frac{1}{k} - \ln n$$

is bounded. Moreover,

$$a_n - a_{n+1} = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n+1) = \ln\big(1 + \frac{1}{n}\big) - \frac{1}{n+1}\,.$$

Since the derivative of the function $f(x) = \ln(1+x) - \dfrac{x}{x+1}$ is positive on $[0,1]$, we find that f is increasing on $[0,1]$; thus

$$\ln\big(1 + \frac{1}{n}\big) - \frac{1}{n+1} = f\big(\frac{1}{n}\big) \geqslant f(0) = \ln 1 - \frac{0}{1} = 0 \qquad \forall\, n \in \mathbb{N}$$

which shows that $a_n \geqslant a_{n+1}$. Therefore, $\{a_n\}_{n=1}^{\infty}$ is monotone decreasing and bounded from below (by 0). The completeness of real numbers then implies the convergence of the sequence $\{a_n\}_{n=1}^{\infty}$. The limit

$$\lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \ln n \Big)$$

is called Euler's constant. Euler's constant is approximated 0.5772.

9.3.3 Error estimates

Similar to (9.3.1), under the same setting we have

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant S \leqslant S_n + \int_{n}^{\infty} f(x)\,dx \qquad \forall\, n \in \mathbb{N}\,. \tag{9.3.2}$$

The inequality above shows the following

Theorem 9.37: Bounds for the Remainder in the Integral Test

Let $f : [1,\infty) \to \mathbb{R}$ be a non-negative continuous decreasing function such that the series $S = \sum\limits_{k=1}^{\infty} f(k)$ converges. Then the remainder $R_n = S - S_n$, where $S_n = \sum\limits_{k=1}^{n} f(k)$, satisfies the inequality

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx\,.$$

Example 9.38. Estimate the sum of the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ using the inequalities in (9.3.2) and $n = 10$.

Since

$$\int_{n}^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \frac{-1}{x}\Big|_{x=n}^{x=b} = \frac{1}{n}\,,$$

using (9.3.2) we find that

$$S_{10} + \frac{1}{11} \leqslant \sum_{k=1}^{\infty} \frac{1}{k^2} \leqslant S_{10} + \frac{1}{10}\,.$$

Computing S_{10}, we obtain that

$$S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{81} + \frac{1}{100} \approx 1.54977\,;$$

thus

$$1.64068 \leqslant \sum_{k=1}^{\infty} \frac{1}{k^2} \leqslant 1.64977\,.$$

9.4 Comparisons of Series

When the sequence $\{a_n\}_{n=1}^{\infty}$ is not obtained by $a_n = f(n)$ for some decreasing function $f : [1,\infty) \to \mathbb{R}$, the convergence of the series $\sum\limits_{k=1}^{\infty} a_k$ cannot be judged by the convergence of the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$. To determine the convergence of this kind of series, usually one uses comparison tests.

9.4.1 Direct Comparison Test

Theorem 9.39

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$.

1. If $\sum\limits_{k=1}^{\infty} b_k$ converges, then $\sum\limits_{k=1}^{\infty} a_k$ converges.
2. If $\sum\limits_{k=1}^{\infty} a_k$ diverges, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Proof. Let S_n and T_n be the n-th partial sum of the series $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$, respectively; that is,

$$S_n = \sum_{k=1}^{n} a_k \qquad \text{and} \qquad T_n = \sum_{k=1}^{n} b_k\,.$$

Then by the assumption that $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$, we find that $0 \leqslant S_n \leqslant T_n$ for all $n \in \mathbb{N}$, and $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are monotone increasing sequences.

1. If $\sum_{k=1}^{\infty} b_k$ converges, $\lim_{n\to\infty} T_n = T$ exists; thus $0 \leqslant S_n \leqslant T_n \leqslant T$ for all $n \in \mathbb{N}$. Since $\{S_n\}_{n=1}^{\infty}$ is increasing, the monotone sequence property shows that $\lim_{n\to\infty} S_n$ exists; thus $\sum_{k=1}^{\infty} a_k$ converges.

2. If $\sum_{k=1}^{\infty} a_k$ diverges, $\lim_{n\to\infty} S_n = \infty$; thus by the fact that $S_n \leqslant T_n$ for all $n \in \mathbb{N}$, we find that $\lim_{n\to\infty} T_n = \infty$. Therefore, $\sum_{k=1}^{\infty} b_k$ diverges (to ∞). □

Remark 9.40. It does not require that $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$ for the direct comparison test to hold. The condition can be relaxed by that "$0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$" for some N since the sum of the first $N-1$ terms does not affect the convergence of the series.

Example 9.41. The series $\sum_{k=1}^{\infty} \frac{1+\sin k}{k^2}$ converges since $\frac{1+\sin n}{n^2} \leqslant \frac{2}{n^2}$ for all $n \in \mathbb{N}$ and the p-series $\sum_{k=1}^{\infty} \frac{2}{k^2}$ converges.

Example 9.42. The series $\sum_{k=1}^{\infty} \frac{1}{2+3^k}$ converges since $\frac{1}{2+3^n} \leqslant \frac{1}{3^n}$ for all $n \in \mathbb{N}$ and the geometric series $\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges.

Example 9.43. The series $\sum_{k=1}^{\infty} \frac{1}{2+\sqrt{k}}$ diverges since $\frac{1}{2+\sqrt{n}} \geqslant \frac{1}{3\sqrt{n}}$ for all $n \in \mathbb{N}$ and the p-series $\sum_{k=1}^{\infty} \frac{1}{3\sqrt{k}} = \frac{1}{3}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

One can also use the fact that $\frac{1}{2+\sqrt{n}} \geqslant \frac{1}{n}$ for all $n \geqslant 4$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to conclude that $\sum_{k=1}^{\infty} \frac{1}{2+\sqrt{k}}$ diverges.

9.4.2 Limit Comparison Test

Theorem 9.44

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, $a_n, b_n > 0$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L\,,$$

where L is a non-zero real number. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

Proof. We first note that if $L \neq 0$, then $L > 0$ since $\dfrac{a_n}{b_n} > 0$ for all $n \in \mathbb{N}$. By the fact that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$, there exists $N > 0$ such that $\left|\dfrac{a_n}{b_n} - L\right| < \dfrac{L}{2}$ whenever $n \geqslant N$. In other words, $\dfrac{L}{2} < \dfrac{a_n}{b_n} < \dfrac{3L}{2}$ for all $n \geqslant N$; thus

$$0 < a_n < \frac{3L}{2} b_n \quad \text{and} \quad 0 < b_n < \frac{2}{L} a_n \qquad \text{whenever} \quad n \geqslant N\,.$$

By Theorem 9.39 and Remark 9.40, we find that $\sum\limits_{k=1}^{\infty} a_k$ converges if and only if $\sum\limits_{k=1}^{\infty} b_k$ converges. □

Remark 9.45. 1. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then the convergence of $\sum\limits_{k=1}^{\infty} b_k$ implies the convergence of $\sum\limits_{k=1}^{\infty} a_k$, but not necessary the reverse direction.

2. The condition "$a_n, b_n > 0$ for all $n \in \mathbb{N}$" can be relaxed by "a_n and b_n are sign-definite for $n \geqslant N$, where a sequence $\{c_n\}_{n=1}^{\infty}$ is called sign-definite for $n \geqslant N$ if $c_n > 0$ for all $n \geqslant N$ or $c_n < 0$ for all $n \geqslant N$.

Example 9.46. Recall that in Example 9.42 and 9.43 we have shown that the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+3^k}$ converges and the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+\sqrt{k}}$ diverges using the direct comparison test. Note that since

$$\lim_{n\to\infty} \frac{\dfrac{1}{2+3^n}}{\dfrac{1}{3^n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\dfrac{1}{2+\sqrt{n}}}{\dfrac{1}{\sqrt{n}}} = 1\,,$$

using the convergence of the p-series and the limit comparison test we can also conclude that $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+3^k}$ converges and $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+\sqrt{k}}$ diverges.

Example 9.47. The general harmonic series $\sum\limits_{k=1}^{\infty} \dfrac{1}{ak+b}$ diverges for the following reasons:

1. if $a = 0$, then clearly $\sum\limits_{k=1}^{\infty} \dfrac{1}{b}$ diverges.

2. if $a \neq 0$, then $\sum\limits_{k=1}^{\infty} \dfrac{1}{ak}$ diverges and $\lim\limits_{n\to\infty} \dfrac{\dfrac{1}{ak}}{\dfrac{1}{ak+b}} = 1$.

9.5 The Ratio and Root Tests

9.5.1 The Ratio Test

Theorem 9.48: Ratio Test

Let $\sum\limits_{k=1}^{\infty} a_k$ be a series with positive terms.

1. The series $\sum\limits_{k=1}^{\infty} a_k$ converges if $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$.
2. The series $\sum\limits_{k=1}^{\infty} a_k$ diverges (to ∞) if $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$.

Proof. Suppose that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$ exists. Define $r = \dfrac{L+1}{2}$.

1. Assume that $L < 1$. Then for $\varepsilon = \dfrac{1-L}{2}$, there exists $N > 0$ such that
$$\Big|\frac{a_{n+1}}{a_n} - L\Big| < \frac{1-L}{2} \qquad \text{whenever } n \geqslant N\,;$$
thus
$$0 < \frac{a_{n+1}}{a_n} < r \qquad \text{whenever } n \geqslant N\,.$$
Note that $0 < r < 1$, and the inequality above implies that if $n \geqslant N$, $a_{n+1} < ra_n$. Therefore,
$$0 < a_n \leqslant a_N r^{n-N} \qquad \text{for all } n \geqslant N\,.$$
Now, since the series $\sum\limits_{k=1}^{\infty} a_N r^k$ converges, the comparison test implies that $\sum\limits_{k=1}^{\infty} a_k$ converges as well.

2. Assume that $L > 1$. Then for $\varepsilon = \dfrac{L-1}{2}$, there exists $N > 0$ such that
$$\Big|\frac{a_{n+1}}{a_n} - L\Big| < \frac{L-1}{2} \qquad \text{whenever } n \geqslant N\,;$$
thus
$$r < \frac{a_{n+1}}{a_n} \qquad \text{whenever } n \geqslant N\,.$$
Note that $r > 1$, and the inequality above implies that if $n \geqslant N$, $a_{n+1} > ra_n$. Therefore,
$$0 < a_N r^{n-N} \leqslant a_n \qquad \text{for all } n \geqslant N\,.$$

Now, since the series $\sum_{k=1}^{\infty} a_N r^{k-N}$ diverges, the comparison test implies that $\sum_{k=1}^{\infty} a_k$ diverges as well. □

Remark 9.49. When $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the convergence or divergence of $\sum_{n=1}^{\infty} a_k$ cannot be concluded. For example, the p-series could converge or diverge depending on how large p is, but no matter what p is,

$$\lim_{n\to\infty} \frac{(n+1)^p}{n^p} = 1\,.$$

Example 9.50. The series $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges since

$$\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1\,.$$

Example 9.51. The series $\sum_{k=1}^{\infty} \frac{k^2 2^{k+1}}{3^k}$ converges since

$$\lim_{n\to\infty} \frac{(n+1)^2 2^{n+2}/3^{n+1}}{n^2 2^{n+1}/3^n} = \lim_{n\to\infty} \frac{2}{3}\frac{(n+1)^2}{n^2} = \frac{2}{3} < 1\,.$$

Example 9.52. The series $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ diverges since

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1\,.$$

9.5.2 The Root Test

Theorem 9.53: Root Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms.

1. The series $\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$.
2. The series $\sum_{k=1}^{\infty} a_k$ diverges (to ∞) if $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$.

Proof. Suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ exists. Define $r = \frac{L+1}{2}$.

1. Assume that $L < 1$. Then for $\varepsilon = \dfrac{1-L}{2}$, there exists $N > 0$ such that
$$\left|\sqrt[n]{a_n} - L\right| < \frac{1-L}{2} \qquad \text{whenever } n \geqslant N\,;$$
thus
$$0 < \sqrt[n]{a_n} < r \qquad \text{whenever } n \geqslant N$$
or equivalently,
$$0 < a_n \leqslant r^n \qquad \text{whenever } n \geqslant N\,.$$
By the fact that $0 < r < 1$, the series $\sum\limits_{k=1}^{\infty} r^k$ converges; thus the comparison test implies that $\sum\limits_{k=1}^{\infty} a_k$ converges as well.

2. Left as an exercise. □

Remark 9.54. When $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$, the convergence or divergence of $\sum\limits_{n=1}^{\infty} a_k$ cannot be concluded. For example, the p-series could converge or diverge depending on how large p is, but no matter what p is,
$$\lim_{n\to\infty} \sqrt[n]{n^p} = \Big(\lim_{n\to\infty} \sqrt[n]{n}\Big)^p = 1\,.$$

Example 9.55. The series $\sum\limits_{k=1}^{\infty} \dfrac{e^{2k}}{k^k}$ converges since
$$\lim_{n\to\infty} \Big(\frac{e^{2n}}{n^n}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{e^2}{n} = 0 < 1\,.$$
We also note that the convergence of this series can be obtained through the ratio test:
$$\lim_{n\to\infty} \frac{e^{2(n+1)}/(n+1)^{n+1}}{e^{2n}/n^n} = \lim_{n\to\infty} \frac{e^2}{n+1}\Big(1+\frac{1}{n}\Big)^{-n} = 0 < 1\,.$$

Example 9.56. The series $\sum\limits_{k=1}^{\infty} \dfrac{k^2 2^{k+1}}{3^k}$ converges since
$$\lim_{n\to\infty} \Big(\frac{n^2 2^{n+1}}{3^n}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2(2n^2)^{\frac{1}{n}}}{3} = \frac{2}{3} < 1\,.$$

Example 9.57. The series $\sum\limits_{k=1}^{\infty} \dfrac{k^k}{k!}$ diverges since
$$\lim_{n\to\infty} \Big(\frac{n^n}{n!}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \Big(\frac{n^n}{\sqrt{2\pi n}n^n e^{-n}} \frac{\sqrt{2\pi n}n^n e^{-n}}{n!}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \Big(\frac{e^n}{\sqrt{2\pi n}}\Big)^{\frac{1}{n}} = e > 1\,,$$
here we have used Stirling's formula (9.1.2) to compute the limit.

Remark 9.58. Observe from Example 9.51, 9.52, 9.56 and 9.57, we see that as long as $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists, then the limits are the same. This is in fact true in general, but we will not prove it since this is not our focus.

9.6 Absolute and Conditional Convergence

In the previous three sections we consider the convergence of series whose terms do not have different signs. How about the convergence of series like

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^p}, \qquad \sum_{k=1}^{\infty} \frac{\sin k}{k^p} \qquad \text{and etc.}$$

In the following two sections, we will focus on how to judge the convergence of a series that has both positive and negative terms.

Definition 9.59

An infinite series $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent or converge absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges. An infinite series $\sum_{k=1}^{\infty} a_k$ is said to be conditionally convergent or converge conditionally if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges (to ∞).

Example 9.60. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$ converge absolutely for $p > 1$ but does not converge absolutely for $p \leqslant 1$ since the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leqslant 1$.

Example 9.61. The series $\sum_{k=1}^{\infty} \frac{\sin k}{k^p}$ converges absolutely for $p > 1$ since

$$0 \leqslant \left|\frac{\sin n}{n^p}\right| \leqslant \frac{1}{n^p} \qquad \forall\, n \in \mathbb{N}$$

and the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$.

Theorem 9.62

An absolutely convergent series is convergent.（絕對收斂則收斂）

Proof. Let $\sum\limits_{k=1}^{\infty} a_k$ be an absolutely convergent series, and $\varepsilon > 0$ be given. Since $\sum\limits_{k=1}^{\infty} |a_k|$ converges, the Cauchy criteria implies that there exists $N > 0$ such that

$$\Big|\sum_{k=n}^{n+p} |a_k|\Big| < \varepsilon \qquad \text{whenever } n \geqslant N \text{ and } p \geqslant 0\,.$$

Therefore, if $n \geqslant N$ and $p \geqslant 0$,

$$\Big|\sum_{k=n}^{n+p} a_k\Big| \leqslant \sum_{k=n}^{n+p} |a_k| < \varepsilon$$

thus the Cauchy criteria implies that $\sum\limits_{k=1}^{\infty} a_k$ converges. □

Corollary 9.63: Ratio and Root Tests

The series $\sum\limits_{k=1}^{\infty} a_k$ converges if $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} < 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

Example 9.64. The series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k 2^k}{k!}$ converges since

$$\lim_{n\to\infty} \frac{\Big|\dfrac{(-1)^{n+1}2^{n+1}}{(n+1)!}\Big|}{\Big|\dfrac{(-1)^n 2^n}{n!}\Big|} = \lim_{n\to\infty} \frac{\dfrac{2^{n+1}}{(n+1)!}}{\dfrac{2^n}{n!}} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

which shows the absolute convergence of the series the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k 2^k}{k!}$.

Example 9.65. The series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1} k!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)}$ converges since

$$\lim_{n\to\infty} \frac{\Big|\dfrac{(-1)^{n+2}(n+1)!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+3)}\Big|}{\Big|\dfrac{(-1)^{n+1}n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)}\Big|} = \lim_{n\to\infty} \frac{\dfrac{(n+1)!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+3)}}{\dfrac{n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)}} = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} < 1$$

which shows the absolute convergence of the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1} k!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)}$.

Example 9.66. Consider the series $\sum_{k=1}^{\infty} \frac{(k^2 \sin k)^k}{(k!)^k}$. Since

$$\lim_{n\to\infty} \left[\frac{n^{2n}}{(n!)^n}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n!} = \lim_{n\to\infty} \frac{n}{n-1}\frac{1}{(n-2)!} = 0 < 1\,,$$

the series $\sum_{k=1}^{\infty} \frac{k^{2k}}{(k!)^k}$ converges absolutely. By the fact that

$$\left|\frac{(n^2 \sin n)^n}{(n!)^n}\right| \leqslant \frac{(n^2)^n}{(n!)^n} \qquad \forall\, n \in \mathbb{N}\,,$$

the comparison test implies that the series $\sum_{k=1}^{\infty} \frac{(k^2 \sin k)^k}{(k!)^k}$ converges absolutely.

9.6.1 Alternating Series

In the previous two sections we consider the convergence of series whose terms do not have different signs. How about the convergence of series like

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\,, \qquad \sum_{k=1}^{\infty} \frac{\sin k}{k} \quad \text{and etc.}$$

In the following two sections, we will focus on how to judge the convergence of a series that has both positive and negative terms.

Theorem 9.67: Dirichlet's Test

Let $\{a_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ be sequences of real numbers such that

1. the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$ is bounded; that is, there exists $M \in \mathbb{R}$ such that $\left|\sum_{k=1}^{n} a_k\right| \leqslant M$ for all $n \in \mathbb{N}$.
2. $\{p_n\}_{n=1}^{\infty}$ is a decreasing sequence, and $\lim_{n\to\infty} p_n = 0$.

Then $\sum_{k=1}^{\infty} a_k p_k$ converges.

Proof. Let $\varepsilon > 0$ be given. Since $\{p_n\}_{n=1}^{\infty}$ is decreasing and $\lim_{n\to\infty} p_n = 0$, there exists $N > 0$ such that

$$0 \leqslant p_n < \frac{\varepsilon}{2M+1} \qquad \text{whenever} \quad n \geqslant N\,.$$

Define $S_n = \sum\limits_{k=1}^{n} a_k$. Then if $n \geqslant N$ and $\ell \geqslant 0$,

$$\begin{aligned}
\Big|\sum_{k=n}^{n+\ell} a_k p_k\Big| &= \big|(S_n - S_{n-1})p_n + (S_{n+1} - S_n)p_{n+1} + (S_{n+2} - S_{n+1})p_{n+2} + \cdots \\
&\quad + (S_{n+\ell-1} - S_{n+\ell-2})p_{n+\ell-1} + (S_{n+\ell} - S_{n+\ell-1})p_{n+\ell}\big| \\
&= \big|-S_{n-1}p_n + S_n(p_n - p_{n+1}) + S_{n+1}(p_{n+1} - p_{n+2}) + \cdots + S_{n+\ell-1}(p_{n+\ell-1} - p_{n+\ell}) \\
&\quad + S_{n+\ell}p_{n+\ell}\big| \\
&\leqslant |S_{n-1}p_n| + |S_n(p_n - p_{n+1})| + |S_{n+1}(p_{n+1} - p_{n+2})| + \cdots + |S_{n+\ell}(p_{n+\ell-1} - p_{n+\ell})| \\
&\quad + |S_{n+\ell+1}p_{n+\ell}| \\
&\leqslant Mp_n + M(p_n - p_{n+1}) + M(p_{n+1} - p_{n+2}) + \cdots + M(p_{n+\ell-1} - p_{n+\ell}) + Mp_{n+\ell} \\
&= 2Mp_n < \frac{2M\varepsilon}{2M+1} < \varepsilon\,.
\end{aligned}$$

The convergence of $\sum\limits_{k=1}^{\infty} a_k p_k$ then follows from the Cauchy criteria (Theorem 9.27). □

Corollary 9.68

Let $\{p_n\}_{n=1}^{\infty}$ be a decreasing sequence of real numbers. If $\lim\limits_{n\to\infty} p_n = 0$, then $\sum\limits_{k=1}^{\infty}(-1)^k p_k$ and $\sum\limits_{k=1}^{\infty}(-1)^{k+1}p_k$ converge.

Example 9.69. The series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^p}$ converges conditionally for $0 < p \leqslant 1$ since

1. $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^p}$ converges due the fact that

$$\Big|\sum_{k=1}^{n}(-1)^{k+1}\Big| \leqslant 1 \quad \text{and} \quad \Big\{\frac{1}{n^p}\Big\}_{n=1}^{\infty} \text{ is decreasing and converges to } 0\,.$$

2. $\sum\limits_{k=1}^{\infty} \Big|\dfrac{(-1)^{k+1}}{k^p}\Big|$ diverges for it is a p-series with $0 < p \leqslant 1$.

Similarly, $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{\ln(k+1)}$ converges conditionally.

Example 9.70. The series $\sum\limits_{k=1}^{\infty} \dfrac{\sin k}{k^p}$ converges for $p > 0$ since

1. $\sum_{k=1}^{n} \sin k = \dfrac{\cos\frac{1}{2} - \cos\frac{2k+1}{2}}{2\sin\frac{1}{2}}$; $\Big($thus $\Big|\sum_{k=1}^{n} \sin k\Big| \leqslant \dfrac{1}{\sin\frac{1}{2}}\Big)$.

2. $\{\frac{1}{n^p}\}_{n=1}^{\infty}$ is decreasing and $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.

We remark here that $\sum_{k=1}^{\infty} \dfrac{\sin k}{k} = \dfrac{\pi - 1}{2}$. In fact, $\sum_{k=1}^{\infty} \dfrac{\sin(kx)}{k}$ is the Fourier series of the function $\dfrac{\pi - x}{2}$.

- **Alternating Series Remainder**

Theorem 9.71

Let $\{a_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ be sequences of real numbers satisfying conditions in Theorem 9.67. If $\Big|\sum_{k=1}^{n} a_k\Big| \leqslant M$ for all $n \in \mathbb{N}$, then

$$\Big|\sum_{k=1}^{\infty} a_k p_k - \sum_{k=1}^{n} a_k p_k\Big| = \Big|\sum_{k=n+1}^{\infty} a_k p_k\Big| \leqslant 2M p_{n+1}\,.$$

Moreover, if $a_k = (-1)^k$, then

$$\Big|\sum_{k=1}^{\infty}(-1)^{k+1} p_k - \sum_{k=1}^{n}(-1)^{k+1} p_k\Big| \leqslant p_{n+1} \qquad \forall\, n \in \mathbb{N}\,.$$

Sketch of Proof. Let $S_n = \sum_{k=1}^{n} a_k$. According to the proof of the Abel test, we have

$$\begin{aligned}\Big|\sum_{k=n}^{n+\ell} a_k p_k\Big| \leqslant |S_{n-1}| p_n + |S_n|(p_n - p_{n+1}) + |S_{n+1}|(p_{n+1} - p_{n+2}) + \cdots + |S_{n+\ell}|(p_{n+\ell-1} - p_{n+\ell}) \\ + |S_{n+\ell+1}| p_{n+\ell}\,. \qquad (9.6.1)\end{aligned}$$

Note that for the general case, by the fact that $|S_n| \leqslant M$ for all $n \in \mathbb{N}$ and $\{p_n\}_{n=1}^{\infty}$ is decreasing, we conclude that for all $\ell \geqslant 0$,

$$\Big|\sum_{k=n}^{n+\ell} a_k p_k\Big| \leqslant 2M p_n \qquad \forall\, n \in \mathbb{N}\,;$$

thus if $n \in \mathbb{N}$,

$$\Big|\sum_{k=1}^{\infty} a_k p_k - \sum_{k=1}^{n} a_k p_k\Big| = \lim_{\ell\to\infty}\Big|\sum_{k=1}^{n+1+\ell} a_k p_k - \sum_{k=1}^{n} a_k p_k\Big| = \lim_{\ell\to\infty}\Big|\sum_{k=n+1}^{n+1+\ell} a_k p_k\Big| \leqslant 2M p_{n+1}\,.$$

For the case of alternating series, we note that terms of $\{S_n\}_{n=1}^{\infty}$ are $\{1, 0, 1, 0, 1, \cdots\}$; thus (9.6.1) implies that

$$\Big|\sum_{k=1}^{\infty}(-1)^{k+1}p_k - \sum_{k=1}^{n}(-1)^{k+1}p_k\Big| \leqslant p_{n+1} \qquad \forall\, n \in \mathbb{N}\,. \qquad \square$$

Example 9.72. Approximate the sum of the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k!}$ by its first six terms, we obtain that

$$\sum_{k=1}^{6}(-1)^{k+1}\frac{1}{k!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} \approx 0.63194\,.$$

Moreover, by Theorem 9.71, we find that

$$\Big|\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k!} - \sum_{k=1}^{6}(-1)^{k+1}\frac{1}{k!}\Big| \leqslant \frac{1}{7!} = \frac{1}{5040} \approx 0.0002\,.$$

Example 9.73. Determine the number of terms required to approximate the sum of the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4}$ with an error of less than 0.0001.

By Theorem 9.71,

$$\Big|\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4} - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k^4}\Big| \leqslant \frac{1}{(n+1)^4}\,;$$

thus choosing n such that $\frac{1}{(n+1)^4} \leqslant 0.0001$ (that is, $n \geqslant 9$), we obtain that

$$\Big|\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4} - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k^4}\Big| \leqslant 0.001 \qquad \forall\, n \geqslant 9\,.$$

9.7 Taylor Polynomials and Approximations

Suppose that $f : (a, b) \to \mathbb{R}$ is $(n+1)$-times continuously differentiable; that is, $\frac{d^k f}{dx^k}$ is continuous on (a, b) for $1 \leqslant k \leqslant n+1$, then for $x \in (a, b)$, the Fundamental Theorem of

Calculus and integration-by-parts imply that

$$\begin{aligned}
f(x) - f(c) = \int_c^x f'(t)\,dt &= f'(t)(t-x)\Big|_{t=c}^{t=x} - \int_c^x f''(t)(t-x)\,dt \\
&= -f'(c)(c-x) - \int_c^x f''(t)(t-x)\,dt \\
&= f'(c)(x-c) - \Big[f''(t)\frac{(t-x)^2}{2}\Big|_{t=c}^{t=x} - \int_c^x f'''(t)\frac{(t-x)^2}{2}\,dt\Big] \\
&= f'(c)(x-c) - \Big[-\frac{f''(c)}{2}(c-x)^2 - \int_c^x f'''(t)\frac{(t-x)^2}{2}\,dt\Big] \\
&= f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \int_c^x f'''(t)\frac{(t-x)^2}{2}\,dt \\
&= \cdots\cdots \\
&= f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\
&\quad + (-1)^n \int_c^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt\,,
\end{aligned}$$

where the last equality can be shown by induction. Therefore,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^n \int_c^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt\,. \tag{9.7.1}$$

Definition 9.74

If f has n derivatives at c, then the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is called the n-th (order) Taylor polynomial for f at c. The n-th Taylor polynomial for f at 0 is also called the n-th (order) Maclaurin polynomial for f.

Example 9.75. The n-th Maclaurin polynomial for the function $f(x) = e^x$ is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{n} \frac{1}{k!}x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\,.$$

Example 9.76. The n-th Maclaurin polynomial for the function $f(x) = \ln(1+x)$ is given by

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!} x^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}}{n} x^n \,, \end{aligned}$$

here we have used $g^{(k)}(x) = (-1)^{k-1}(k-1)!(x+1)^{-k}$ to compute $g^{(k)}(0)$.

The n-th Taylor polynomial for the function $g(x) = \ln x$ at 1 is given by

$$\begin{aligned} Q_n(x) &= \sum_{k=0}^{n} \frac{g^{(k)}(1)}{k!} (x-1)^k = \sum_{k=1}^{n} \frac{g^{(k)}(1)}{k!} (x-1)^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!} (x-1)^k \\ &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (x-1)^k \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}}{n} (x-1)^n \,, \end{aligned}$$

here we have used $g^{(k)}(x) = (-1)^{k-1}(k-1)!x^{-k}$ to compute $g^{(k)}(1)$. We note that $Q_n(x) = P_n(x-1)$ (and $g(x) = f(x-1)$).

Example 9.77. The $(2n)$-th Maclaurin polynomial for the function $f(x) = \cos x$ is given by

$$\begin{aligned} P_{2n}(x) &= \sum_{k=0}^{2n} \frac{f^{(k)}(0)}{k!} x^k = 1 + \sum_{k=1}^{2n} \frac{f^{(k)}(0)}{k!} x^k = 1 + \sum_{k=1}^{n} \frac{f^{(2k-1)}(0)}{(2k-1)!} x^{2k-1} + \sum_{k=1}^{n} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} \\ &= 1 + \sum_{k=1}^{n} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} \,, \end{aligned}$$

here we have used $f^{(k)}(x) = \cos\left(x + \frac{k\pi}{2}\right)$ to compute $f^{(k)}(0)$. We also note that $P_{2n}(x) = P_{2n+1}(x)$ for all $n \in \mathbb{N}$.

The $(2n-1)$-th Maclaurin polynomial for the function $g(x) = \sin x$ is given by

$$\begin{aligned} Q_{2n-1}(x) &= \sum_{k=0}^{2n-1} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=1}^{2n-1} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{g^{(2k-1)}(0)}{(2k-1)!} x^{2k-1} + \sum_{k=1}^{n} \frac{g^{(2k)}(0)}{(2k)!} x^{2k} \\ &= \sum_{k=1}^{n} \frac{g^{(2k-1)}(0)}{(2k-1)!} x^{2k-1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1} \,, \end{aligned}$$

here we have used $g^{(k)}(x) = \sin\left(x + \frac{k\pi}{2}\right)$ to compute $g^{(k)}(0)$. We also note that $Q_{2n-1}(x) = Q_{2n}(x)$ for all $n \in \mathbb{N}$.

9.7.1 Remainder of Taylor Polynomials

To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial, we look for the difference $R_n(x) \equiv f(x) - P_n(x)$, where P_n is the n-th Taylor polynomial for f (centered at a certain number c). The function R_n is called the remainder associated with the approximation P_n.

• Integral form of the remainder

By (9.7.1), we find that if P_n is the n-th Taylor polynomial for f at c, then

$$R_n(x) = (-1)^n \int_c^x f^{(n+1)}(t) \frac{(t-x)^n}{n!}\, dt\,. \tag{9.7.2}$$

Example 9.78. Consider the function $f(x) = \exp(x) = e^x$. If P_n is the n-th Maclaurin polynomial for f, the remainder R_n associated with P_n is given by

$$R_n(x) = (-1)^n \int_0^x f^{(n+1)}(t) \frac{(t-x)^n}{n!}\, dt = (-1)^n \int_0^x e^t \frac{(t-x)^n}{n!}\, dt\,.$$

Therefore, if $x > 0$,

$$\Big| e^x - \sum_{k=0}^n \frac{x^k}{k!} \Big| = \Big| \int_0^x e^t \frac{(t-x)^n}{n!}\, dt \Big| \leqslant \int_0^x e^t \frac{(x-t)^n}{n!}\, dt \leqslant \int_0^x e^x \frac{x^n}{n!}\, dt = \frac{e^x x^{n+1}}{n!}\,. \tag{9.7.3}$$

Note that for each $x > 0$, the series $\sum\limits_{k=0}^{\infty} e^x \dfrac{x^{n+1}}{n!}$ converges since

$$\lim_{n\to\infty} \frac{e^x \dfrac{x^{(n+1)+1}}{(n+1)!}}{e^x \dfrac{x^{n+1}}{n!}} = \lim_{n\to\infty} \frac{x}{n+1} = 0\,;$$

thus the n-th term test shows that $\lim\limits_{n\to\infty} e^x \dfrac{x^{n+1}}{n!} = 0$. Therefore, for each $x > 0$,

$$\lim_{n\to\infty} \Big| e^x - \sum_{k=0}^n \frac{x^k}{k!} \Big| = 0$$

or equivalently,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\,.$$

In particular, if $x = 1$, (9.7.3) implies that

$$\Big| e - \sum_{k=0}^{n} \frac{1}{k!} \Big| \leqslant \frac{e}{n!} \,;$$

thus $\Big| e - \sum\limits_{k=0}^{17} \frac{1}{k!} \Big| < 10^{-8}$.

Example 9.79. Consider the function $f(x) = \cos x$ and its $(2n)$-th Maclaurin polynomial P_{2n} in Example 9.77. If $x > 0$,

$$\begin{aligned} \big| f(x) - P_{2n}(x) \big| = \big| f(x) - P_{2n+1}(x) \big| &\leqslant \Big| \int_0^x f^{(2n+2)}(t) \frac{(t-x)^{2n+1}}{(2n+1)!} \, dt \Big| \leqslant \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} \, dt \\ &= \frac{-(x-t)^{2n+2}}{(2n+2)!} \Big|_{t=0}^{t=x} = \frac{x^{2n+2}}{(2n+2)!} \,, \end{aligned}$$

while if $x < 0$,

$$\begin{aligned} \big| f(x) - P_{2n}(x) \big| = \big| f(x) - P_{2n+1}(x) \big| &\leqslant \Big| \int_0^x f^{(2n+2)}(t) \frac{(t-x)^{2n+1}}{(2n+1)!} \, dt \Big| \leqslant \int_x^0 \frac{(t-x)^{2n+1}}{(2n+1)!} \, dt \\ &= \frac{(t-x)^{2n+2}}{(2n+2)!} \Big|_{t=0}^{t=x} = \frac{(-x)^{2n+2}}{(2n+2)!} \,. \end{aligned}$$

Therefore,

$$\Big| \cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} \Big| \leqslant \frac{|x|^{2n+2}}{(2n+2)!} \qquad \forall \, x \in \mathbb{R} \,. \tag{9.7.4}$$

Similarly,

$$\Big| \sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \Big| \leqslant \frac{|x|^{2n+3}}{(2n+3)!} \qquad \forall \, x \in \mathbb{R} \,. \tag{9.7.5}$$

Moreover, by the fact that

$$\lim_{n\to\infty} \frac{\dfrac{|x|^{2(n+1)+2}}{[2(n+1)+2]!}}{\dfrac{|x|^{2n+2}}{(2n+2)!}} = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+4)} = 0 < 1$$

and

$$\lim_{n\to\infty} \frac{\dfrac{|x|^{2(n+1)+3}}{[2(n+1)+3]!}}{\dfrac{|x|^{2n+3}}{(2n+3)!}} = \lim_{n\to\infty} \frac{x^2}{(2n+4)(2n+5)} = 0 < 1$$

the ratio test implies that $\sum\limits_{k=0}^{\infty} \frac{|x|^{2n+2}}{(2n+2)!}$ and $\sum\limits_{k=0}^{\infty} \frac{|x|^{2n+3}}{(2n+3)!}$ converge; thus for each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} = \lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} = 0\,;$$

thus

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots\,,$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots\,.$$

Using (9.7.4), we conclude that

$$\Big| \cos(0.1) - \sum_{k=0}^{3} \frac{(-1)^k}{(2k)!} (0.1)^{2k} \Big| \leqslant \frac{0.1^8}{8!}\,;$$

thus $\cos(0.1) \approx \sum\limits_{k=0}^{3} \frac{(-1)^k}{(2k)!} (0.1)^{2k} \approx 0.995004165$ which is accurate to nine decimal points.

Remark 9.80. By Example 9.78 and 9.79, conceptually we can explain why the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$ for all $\theta \in \mathbb{R}$. Recall that the $(2n)$-th Maclaurin polynomial for exp, cos, sin are

$$P^e_{2n}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!}\,,$$
$$P^c_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n}\,,$$
$$P^s_{2n}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1}\,.$$

Substitution $x = i\theta$, we find that

$$P^e_{2n}(i\theta) = P^c_{2n}(\theta) + iP^s_{2n}(\theta) \qquad \forall\, \theta \in \mathbb{R}\,.$$

Passing $n \to \infty$, by the fact that the remainders $R_n(x)$ for exp, sin and cos all converges to zero as $n \to \infty$ for each $x \in \mathbb{R}$ (and even $x \in \mathbb{C}$), we conclude that

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \forall\, \theta \in \mathbb{R}\,.$$

- **Lagrange form of the remainder**

Theorem 9.81: Taylor's Theorem

Let $f : (a,b) \to \mathbb{R}$ be $(n+1)$-times differentiable, and $c \in (a,b)$. Then for each $x \in (a,b)$, there exists ξ between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)\,, \quad (9.7.6)$$

where Lagrange form of the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\,.$$

Proof. We first show that if $h : (a,b) \to \mathbb{R}$ is m-times differentiable, and $c \in (a,b)$. Then for all $d \in (a,b)$ and $d \neq c$ there exists ξ between c and d such that

$$\frac{h(d) - \sum\limits_{k=0}^{m} \frac{h^{(k)}(c)}{k!}(d-c)^k}{(d-c)^{m+1}} = \frac{1}{m+1}\,\frac{h'(\xi) - \sum\limits_{k=0}^{m-1} \frac{(h')^{(k)}(c)}{k!}(\xi-c)^k}{(\xi-c)^m}\,. \quad (9.7.7)$$

Let $F(x) = h(x) - \sum\limits_{k=0}^{m} \frac{h^{(k)}(c)}{k!}(x-c)^k$ and $G(x) = (x-c)^m$. Then F, G are continuous on $[c,d]$ (or $[d,c]$) and differentiable on (c,d) (or (d,c)), and $G'(x) \neq 0$ for all $x \neq c$. Therefore, the Cauchy Mean Value Theorem implies that there exists ξ between c and d such that

$$\frac{F(d) - F(c)}{G(d) - G(c)} = \frac{F'(\xi)}{G'(\xi)}\,,$$

and (9.7.7) is exactly the explicit form of the equality above.

Now we apply (9.7.7) successfully for $h = f,\ f',\ f'',\ \cdots$ and $f^{(n)}$ and find that

$$\begin{aligned}
\frac{f(d) - \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(d-c)^k}{(d-c)^{n+1}} &= \frac{1}{n+1}\,\frac{f'(d_1) - \sum\limits_{k=0}^{n-1} \frac{(f')^{(k)}(c)}{k!}(d_1-c)^k}{(d_1-c)^n} \\
&= \frac{1}{n+1}\cdot\frac{1}{n}\,\frac{f''(d_2) - \sum\limits_{k=0}^{n-2} \frac{(f'')^{(k)}(c)}{k!}(d_2-c)^k}{(d_2-c)^{n-1}} \\
&= \cdots\cdots \\
&= \frac{1}{(n+1)!}\,\frac{f^{(n)}(d_n) - f^{(n)}(c)}{d_n - c} = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\,;
\end{aligned}$$

thus

$$f(d) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(d-c)^k = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(d-c)^{n+1} .$$

(9.7.6) then follows from the equality above since $d \in (a, b)$ is given arbitrary. □

Example 9.82. In Example 9.76 we compute the Taylor polynomial Q_n for the function $y = \ln(1 + x)$. Note that the Taylor Theorem implies that

$$\ln(1 + x) = P_n(x) + R_n(x) ,$$

where

$$R_n(x) = \frac{1}{(n+1)!}\Big(\frac{d^{n+1}}{dx^{n+1}}\Big|_{x=\xi} \ln(1 + x)\Big)x^{n+1} = \frac{(-1)^n}{n+1}(1 + \xi)^{-n-1}x^{n+1}$$

for some ξ between 0 and x.

1. If $-1 < x < 0$, then $R_n(x) = \dfrac{-1}{n+1}\Big(\dfrac{-x}{1+\xi}\Big)^{n+1} < 0$; thus

$$\ln(1 + x) \leqslant x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n}{n} x^n \qquad \forall\, x \in (-1, 0) \text{ and } n \in \mathbb{N} .$$

2. If $x > 0$, then

   (a) $R_n(x) < 0$ if n is odd; thus

$$\ln(1 + x) \leqslant x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{1}{2k+1} x^{2k+1} \qquad \forall\, x > 0 \text{ and } k \in \mathbb{N} .$$

   (b) $R_n(x) > 0$ if n is even; thus

$$\ln(1 + x) \geqslant x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{-1}{2k} x^{2k} \qquad \forall\, x > 0 \text{ and } k \in \mathbb{N} .$$

Example 9.83. In this example we show that

$$\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots \qquad \forall\, x \in (0, 1] . \quad (9.7.8)$$

Note that Taylor's Theorem implies that for all $x > -1$, there exists ξ between 0 and x such that the remainder associated with $P_n(x) = \sum\limits_{k=1}^{n} \dfrac{(-1)^{k-1}x^k}{k}$ is given by

$$R_n(x) = \frac{(-1)^n}{n+1}(1 + \xi)^{-n-1}x^{n+1} .$$

Note that since ξ is between 0 and x, we always have

$$0 < \frac{x}{1+\xi} < 1 \qquad \forall\, x \in (0,1]\,;$$

thus $|R_n(x)| \leqslant \dfrac{1}{n+1}$ for all $x \in (-1,1]$ and (9.7.8) is concluded because

$$\lim_{n\to\infty} \big|R_n(x)\big| = 0\,.$$

Example 9.84. In this example we compute $\ln 2$. Note that using (9.7.8) we find that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + R_n(1)\,,$$

where

$$R_n(1) = \frac{1}{(n+1)!}\Big(\frac{d^{n+1}}{dx^{n+1}}\Big|_{x=\xi} \ln(1+x)\Big) 1^{n+1} = \frac{(-1)^n}{n+1}(1+\xi)^{-(n+1)}$$

for some ξ between 0 and 1. Since ξ could be very closed to 0, in this case the best we can estimate $R_n(1)$ is

$$\big|R_n(1)\big| \leqslant \frac{1}{n+1}\,.$$

Therefore, to evaluate $\ln 2$ accurate to eight decimal point, it is required that $n = 10^8$.

Let $c = \dfrac{e}{2} \approx 1.359140914$. Then

$$\ln c = \ln\big(1 + (c-1)\big) = (c-1) - \frac{(c-1)^2}{2} + \cdots + \frac{(-1)^{n-1}}{n}(c-1)^n + R_n(c-1)\,,$$

where $R_n(c-1)$ is given by

$$R_n(c-1) = \frac{1}{(n+1)!}\Big(\frac{d^{n+1}}{dx^{n+1}}\Big|_{x=\xi} \ln(1+x)\Big)(c-1)^{n+1} = \frac{(-1)^n}{n+1}(1+\xi)^{-(n+1)}(c-1)^{n+1}$$

for some ξ between 0 and $c-1$. Note that

$$\big|R_n(c)\big| \leqslant \frac{(c-1)^{n+1}}{n+1}\,;$$

thus the value

$$(c-1) - \frac{(c-1)^2}{2} + \frac{(c-1)^3}{3} - \frac{(c-1)^4}{4} + \cdots + \frac{1}{17}(c-1)^{17}$$

to approximate $\ln c$ is accurate to eight decimal points $\big($since $\frac{1}{18}0.4^{18} < 10^{-8}\big)$. On the other hand, we have $\ln 2 = 1 - \ln c$, so the value

$$1 - (c-1) + \frac{(c-1)^2}{2} - \frac{(c-1)^3}{3} + \frac{(c-1)^4}{4} + \cdots - \frac{1}{17}(c-1)^{17}$$

to approximate $\ln 2$ is also accurate to eight decimal points.

9.8 Power Series

Recall that for all $x \in \mathbb{R}$, we have shown that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots ,$$
$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots ,$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots .$$

The identities above show that the functions $y = \exp(x)$, $y = \cos x$, $y = \sin x$ can be defined using series whose terms are multiples of monomials of x. These kind of series are called power series. To be more precise, we have the following

Definition 9.85: Power Series

Let c be a real number. A power series (of one variable x) centered at c is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \cdots ,$$

where a_k is independent of x and represents the coefficient of the k-th term.

Theorem 9.86

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers. If $\sum\limits_{k=0}^{\infty} a_k d^k$ converges, then $\sum\limits_{k=0}^{\infty} a_k (x-c)^k$ converges absolutely for all $x \in (c - |d|, c + |d|)$.

Proof. First we note that since $\sum\limits_{k=0}^{\infty} a_k d^k$ converges, $\lim\limits_{n\to\infty} a_n d^n = 0$; thus the boundedness of convergent sequence implies that there exists $M > 0$ such that

$$|a_n d^n| \leqslant M \qquad \forall\, n \in \mathbb{N} .$$

Suppose that $|x - c| < |d|$. Then there exists $\varepsilon > 0$ such that $|x - c| < |d| - \varepsilon$. Then

$$|a_n||x-c|^n = |a_n||d|^n \frac{|x-c|^n}{(|d|-\varepsilon)^n} \Big(\frac{|d|-\varepsilon}{|d|}\Big)^n \leqslant M \Big(\frac{|d|-\varepsilon}{|d|}\Big)^n .$$

Therefore, by the convergence of geometric series with ratio between -1 and 1, the direct comparison test implies that the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely. $\square$

Corollary 9.87

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists $R > 0$ such that the series converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.
3. The series converges absolutely for all x.

Definition 9.88: Radius of Convergence and Interval of Convergence

Let a power series centered at c be given. If the power series converges only at c, we say that the radius of convergence of the power series is 0. If the power series converges for $|x-c| < R$ but diverges for $|x-c| > R$, we say that the radius of convergence of the power series is R. If the power series converges for all x, we say that the radius of converges of the power series is ∞. The set of all values of x for which the power series converges is called the interval of convergence of the power series.

Remark 9.89. The radius of convergence of a power series centered at c is the greatest lower bound of the set

$$\left\{ r > 0 \,\middle|\, \text{there exists } x \in (c-r, c+r) \text{ such that the power series diverges} \right\}.$$

Example 9.90. Consider the power series $\sum_{k=0}^{\infty} k!x^k$. Note that for each $x \neq 0$,

$$\lim_{k\to\infty} \frac{\left|(k+1)!x^{k+1}\right|}{\left|k!x^k\right|} = \lim_{k\to\infty} (k+1)|x| = \infty\,;$$

thus the ratio test implies that the power series $\sum_{k=0}^{\infty} k!x^k$ diverges for all $x \neq 0$. Therefore, the radius of convergence of $\sum_{k=0}^{\infty} k!x^k$ is 0, and the interval of convergence of $\sum_{k=0}^{\infty} k!x^k$ is $\{0\}$.

Example 9.91. Consider the power series $\sum_{k=0}^{\infty} 3(x-2)^k$. Note that for each $x \in \mathbb{R}$,

$$\lim_{k\to\infty} \frac{3|x-2|^{k+1}}{3|x-2|^k} = \lim_{k\to\infty} |x-2| = |x-2|\,;$$

thus the ratio test implies that the power series $\sum_{k=0}^{\infty} 3(x-2)^k$ converges absolutely if $|x-2| < 1$ and diverges if $|x-2| > 1$. Therefore, the radius of convergence is 1.

To see the interval of convergence, we still need to determine if the power series converges at end-point 1 or 3. However, the power series clearly does not converge at 1 and 3; thus the interval of convergence is $(1,3)$.

Example 9.92. Consider the power series $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$. Note that for each $x \in \mathbb{R}$,

$$\lim_{k\to\infty} \frac{\left|\frac{x^{k+1}}{(k+1)^2}\right|}{\left|\frac{x^k}{k^2}\right|} = \lim_{k\to\infty} \frac{k^2|x|}{(k+1)^2} = |x|\,;$$

thus the ratio test implies that the power series $\sum_{k=0}^{\infty} \frac{x^k}{k^2}$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Therefore, the radius of convergence is 1.

To see the interval of convergence, we note that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges since it is a p-series with $p = 2$, and $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges since it converges absolutely (or simply because it is an alternating series). Therefore, the interval of convergence of the power series is $[-1,1]$.

Example 9.93. Consider the power series $\sum_{k=1}^{\infty} \frac{x^k}{k}$. Note that for each $x \in \mathbb{R}$,

$$\lim_{k\to\infty} \frac{\left|\frac{x^{k+1}}{k+1}\right|}{\left|\frac{x^k}{k}\right|} = \lim_{k\to\infty} \frac{k|x|}{k+1} = |x|\,;$$

thus the ratio test implies that the power series $\sum_{k=0}^{\infty} \frac{x^k}{k}$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Therefore, the radius of convergence is 1.

To see the interval of convergence, we note that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges since it is a p-series with $p = 1$, and $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges since it is an alternating series. Therefore, the interval of convergence of the power series is $[-1,1)$.

Similarly, the power series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k x^k}{k}$ has interval of convergence $(-1, 1]$.

Example 9.94. Consider the power series $\sum\limits_{k=1}^{\infty} \dfrac{x^k}{k^2}$. Note that for each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{\left|\dfrac{x^{n+1}}{(n+1)^2}\right|}{\left|\dfrac{x^n}{n^2}\right|} = \lim_{n\to\infty} \frac{n^2|x|}{(n+1)^2} = |x|\,;$$

thus the ratio test implies that the power series $\sum\limits_{k=1}^{\infty} \dfrac{x^k}{k^2}$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Therefore, the radius of convergence is 1.

To see the interval of convergence, we note that $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2}$ converges since it is a p-series with $p = 2$, and $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{k^2}$ also converges since it converges absolutely (or because of Dirichlet's test). Therefore, the interval of convergence of the power series is $[-1, 1]$.

Remark 9.95. Even though the examples above all has radius of convergence 1, it is not necessary that the radius of convergence of a power series is always 1. For example, the power series $\sum\limits_{k=1}^{\infty} \dfrac{x^k}{2^k k}$ is obtained by replacing x by $\dfrac{x}{2}$ in Example 9.93; thus

$$\sum_{k=1}^{\infty} \frac{x^k}{2^k k} \text{ converges for } \frac{x}{2} \in [-1, 1)$$

or equivalent, the interval of convergence of $\sum\limits_{k=1}^{\infty} \dfrac{x^k}{2^k k}$ is $[-2, 2)$; thus the radius of convergence of this power series is 2.

Example 9.96. The radius of convergence of the power series $\sum\limits_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{(2k+1)!}$ is ∞ since for all $x \in \mathbb{R}$,

$$\lim_{k\to\infty} \frac{\left|\dfrac{(-1)^{k+1} x^{2(k+1)+1}}{[2(k+1)+1]!}\right|}{\left|\dfrac{(-1)^k x^{2k+1}}{(2k+1)!}\right|} = \lim_{k\to\infty} \frac{\left|\dfrac{(-1)^{k+1} x^{2k+3}}{(2k+3)!}\right|}{\left|\dfrac{(-1)^k x^{2k+1}}{(2k+1)!}\right|} = \lim_{k\to\infty} \frac{x^2}{(2k+3)(2k+2)} = 0\,.$$

• Differentiation and Integration of Power Series

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers and $c \in \mathbb{R}$. If the power series $\sum\limits_{k=0}^{\infty} a_k (x-c)^k$ converges in an interval $(c-r, c+r)$, we can ask ourselves whether the function $f : (c-r, c+r)$

defined by $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ is differentiable or not. We note that even though the power series is an infinite sum of differentiable functions (in fact, monomials), it is not clear if the limiting process $\frac{d}{dx}$ commutes with $\sum_{k=0}^{\infty}$ since

$$\lim_{n\to\infty}\lim_{h\to 0} nh^2 = 0 \qquad \text{but} \qquad \lim_{h\to 0}\lim_{n\to\infty} nh^2 = \infty\,.$$

Theorem 9.97: Properties of Functions Defined by Power Series

If the function

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

has a radius of convergence of $R > 0$, then

1. f is differentiable on $(c-R, c+R)$ and

$$f'(x) = \sum_{k=1}^{\infty} k a_k(x-c)^{k-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots\,.$$

2. an anti-derivative of f on $(c-R, c+R)$ is given by

$$\int f(x)\,dx = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-c)^{k+1} = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \cdots\,.$$

The radius of convergence of the power series obtained by differentiating or integrating a power series term by term is the same as the original power series.

Remark 9.98. Theorem 9.97 states that, in many ways, a function defined by a power series behaves like a polynomial; that is, the derivative (or anti-derivative) of a power series can be obtained by term-by-term differentiation (or integration). However, it is not true for general functions defined by series of the form $\sum_{k=0}^{\infty} b_k(x)$. For example, we have talked about (but did not prove) the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ which is the same as $\frac{\pi - x}{2}$ on $(0, 2\pi)$; that is,

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \qquad \forall\, x \in (0, 2\pi)\,.$$

Then

$$-\frac{1}{2} = \frac{d}{dx}\sum_{k=1}^{\infty}\frac{\sin kx}{k} \qquad \forall\, x \in (0, 2\pi)$$

but

$$\frac{d}{dx}\sum_{k=1}^{\infty}\frac{\sin kx}{k} \neq \sum_{k=1}^{\infty}\frac{d}{dx}\frac{\sin kx}{k} = \sum_{k=1}^{\infty}\cos kx \qquad \forall\, x \in (0, 2\pi)$$

since the series $\sum\limits_{k=1}^{\infty}\cos kx$ does not converges for all $x \in (0, 2\pi)$.

Example 9.99. Consider the function f defined by power series

$$f(x) = \sum_{k=1}^{\infty}\frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad \forall\, x \in [-1, 1)\,.$$

Then the function

$$g(x) = \sum_{k=1}^{\infty}x^{k-1} = \sum_{k=0}^{\infty}x^k = 1 + x + x^2 + \cdots\,,$$

obtained by term-by-term differentiation, converges for $x \in (-1, 1)$, and the function

$$h(x) = \sum_{k=1}^{\infty}\frac{x^{k+1}}{k(k+1)} = \sum_{k=2}^{\infty}\frac{x^k}{k(k-1)} = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \cdots$$

obtained by term-by-term differentiation, converges for $x \in [-1, 1]$.

Example 9.100. Suppose that x is a function of t satisfying

$$x''(t) + x(t) = 0\,, \qquad x(0) = x'(0) = 1\,.$$

Assume that $x(t) = \sum\limits_{k=0}^{\infty}a_k t^k$ for $t \in (-R, R)$ with some radius of convergence $R > 0$. Then Theorem 9.97 implies that

$$x''(t) = \sum_{k=2}^{\infty}k(k-1)a_k t^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k \qquad \forall\, t \in (-R, R)\,;$$

thus if $t \in (-R, R)$,

$$\sum_{k=0}^{\infty}\big[(k+2)(k+1)a_{k+2} + a_k\big]t^k = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k + \sum_{k=0}^{\infty}a_k t^k = x''(t) + x(t) = 0\,.$$

The equality above implies that

$$(k+2)(k+1)a_{k+2} + a_k = 0 \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$

Therefore,

$$a_{2k} = \frac{-1}{(2k)(2k-1)} a_{2k-2} = \frac{(-1)^2}{(2k)(2k-1)(2k-2)(2k-4)} a_{2k-4} = \cdots = \frac{(-1)^k}{(2k)!} a_0 \,,$$
$$a_{2k+1} = \frac{-1}{(2k+1)(2k)} a_{2k-1} = \frac{(-1)^2}{(2k+1)(2k)(2k-1)(2k-2)} a_{2k-3} = \cdots = \frac{(-1)^k}{(2k+1)!} a_1 \,.$$

Since $x(0) = x'(0) = 1$ implies $a_0 = a_1 = 1$, we have

$$x(t) = \sum_{k=0}^{\infty} \Big[\frac{(-1)^k}{(2k)!} t^{2k} + \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Big] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} = \cos t + \sin t \,.$$

Corollary 9.101

For a function defined by power series

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$$

(on a certain interval of convergence), the n-th Taylor polynomial for f at c is the n-th partial sum $\sum_{k=0}^{n} a_k (x-c)^k$ of the power series.

9.9 Representation of Functions by Power Series

We have shown the following identities:

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \forall\, x \in \mathbb{R} \,,$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \forall\, x \in \mathbb{R} \,,$$
$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \forall\, x \in \mathbb{R} \,,$$
$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} \qquad \forall\, x \in (-1, 1] \,.$$

In this section, we are interested in finding the power series representation (centered at c) of functions of the form

$$f(x) = \frac{1}{b-x} \,.$$

(without differentiating the function). In other words, for a given $c \in \mathbb{R}\backslash\{b\}$ we would like to find $\{a_k\}_{k=0}^{\infty}$ (which usually depends on c) such that $f(x)$ agrees with the power series

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

on a certain interval of convergence without differentiating f. For example, we know that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \forall\, x \in (-1,1)\,;$$

thus to "expand the function about $\frac{1}{2}$"; that is, to write the function $y = \dfrac{1}{1-x}$ as a power series centered at $\frac{1}{2}$, we have

$$\frac{1}{1-x} = \frac{1}{\frac{1}{2} - \big(x - \frac{1}{2}\big)} = 2 \cdot \frac{1}{1 - 2\big(x - \frac{1}{2}\big)} = 2\sum_{k=0}^{\infty} \Big[2\big(x - \tfrac{1}{2}\big)\Big]^k \text{ if } x \text{ satisfying } 2\big|x - \tfrac{1}{2}\big| < 1\,.$$

In other words, we obtain

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} 2^{k+1}\Big(x - \frac{1}{2}\Big)^k \qquad \forall\, x \in (0,1)$$

without computing the derivatives of the function $y = \dfrac{1}{1-x}$ at $\dfrac{1}{2}$.

We emphasize that f is defined on $\mathbb{R}\backslash\{c\}$ and the power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges only on an interval; thus the function $y = f(x)$ is never the same as the function defined by power series.

- **Geometric Power Series**

Recall that the geometric series $\sum\limits_{k=0}^{\infty} r^k$ converges if and only if $|r| < 1$. The function $g(x) = \dfrac{1}{1-x}$ is defined on $\mathbb{R}\backslash\{1\}$, and by the fact that

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n = \sum_{k=0}^{n} x^k \qquad \forall\, x \neq 1\,,$$

we find that if $|x| < 1$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}\,;$$

thus $\frac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k$ on $(-1,1)$. Therefore, for $c \neq b$,

$$\frac{1}{b-x} = \frac{1}{b-c} \cdot \frac{1}{1-\dfrac{x-c}{b-c}} = \frac{1}{b-c}\sum_{k=0}^{\infty}\Big(\frac{x-c}{b-c}\Big)^k \qquad \forall\, x \text{ satisfying } \Big|\frac{x-c}{b-c}\Big| < 1\,,$$

or equivalently,

$$\frac{1}{b-x} = \sum_{k=0}^{\infty} \frac{1}{(b-c)^{k+1}}(x-c)^k \qquad \forall\, x \in (c-|b-c|, c+|b-c|)\,.$$

Replacing x by $-x$, we find that

$$\frac{1}{b+x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(b-c)^{k+1}}(x+c)^k \qquad \forall\, x \in (-c-|b-c|, -c+|b-c|)\,.$$

Example 9.102. Find a power series representation for $f(x) = \frac{1}{x}$, centered at 1.

To find the power series centered at 1, we rewrite $\frac{1}{x} = \frac{1}{1+(x-1)}$; thus

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{k=0}^{\infty}(1-x)^k = \sum_{k=0}^{\infty}(-1)^k(x-1)^k \qquad \forall\, |x-1| < 1\,.$$

Example 9.103. Find a power series representation for $f(x) = \ln x$ centered at 1.

Note that $\frac{d}{dx}\ln x = \frac{1}{x}$; thus

$$\frac{d}{dx}\ln x = \sum_{k=0}^{\infty}(-1)^k(x-1)^k \qquad \forall\, x \in (0,2)\,.$$

Therefore, by Theorem 9.97,

$$\ln x = C + \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}(x-1)^{k+1} = C + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k \qquad \forall\, x \in (0,2)\,.$$

To determine the constant C, we let $x = 1$ and find that $\ln 1 = C$; thus $C = 0$ and we conclude that

$$\ln x = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k \qquad \forall\, x \in (0,2)\,.$$

We note that the power series converges at $x = 2$, and Example 9.84 shows that

$$\ln 2 = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\,.$$

In other words, the power series $\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k$ is continuous at 2

- **Operations with Power Series**

Let $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-c)^k$ have interval of convergence I_1 and $g(x) = \sum\limits_{k=0}^{\infty} b_k(x-c)^k$ have interval of convergence I_2.

1. $f(\alpha x) = \sum\limits_{k=0}^{\infty} a_k\alpha^k\big(x - \dfrac{c}{\alpha}\big)^k$ on $I \equiv \big\{x \in \mathbb{R} \,\big|\, \alpha x \in I_1\big\}$.

2. $f(x) + g(x) = \sum\limits_{k=0}^{\infty} (a_k + b_k)x^k$ on $I \equiv I_1 \cap I_2$.

3. If $c = 0$ and $N \in \mathbb{N}$, then $f(x^N) = \sum\limits_{k=0}^{\infty} a_k x^{Nk}$ on $I \equiv \big\{x \in \mathbb{R} \,\big|\, x^N \in I_1\big\}$.

4. $f(x)g(x) = \sum\limits_{k=0}^{\infty} d_k(x-c)^k$ on $I \equiv I_1 \cap I_2$, where $d_k = \sum\limits_{j=0}^{k} a_k b_{j-k}$.

Example 9.104. Find a power series for $f(x) = \arctan x$ centered at 0.

Note that $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$; thus

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k} \qquad \forall\, x \in (-1,1)\,.$$

By Theorem 9.97,

$$\arctan x = C + \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1} \qquad \forall\, x \in (-1,1)\,,$$

and the constant C is determined by applying the identity above at $x = 0$; thus $C = \arctan 0$ and

$$\arctan x = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1} \qquad \forall\, x \in (-1,1)\,,$$

We note that the power series converges at $x = \pm 1$. Is it true that $\arctan 1 = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$?

In general, suppose that the function f defined by power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ has a radius of convergence $R > 0$, and g is a continuous function defined on some interval I such that $f(x) = g(x)$ for all $x \in (c-R, c+R) \subsetneq I$. If f is also defined on $c+R$ (or $c-R$), by Theorem 9.97 it is not clear if $\lim\limits_{x\to c+R} f(x) = g(c+R)$ $\big($or $\lim\limits_{x\to c-R} f(x) = g(c-R)\big)$. The following theorem concerns with this issue.

Theorem 9.105: Continuity of Power Series at End-points

Let the radius of convergence of the power series $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ be r for some $r > 0$.

1. If $\sum_{k=0}^{\infty} a_k r^k$ converges, then f is continuous at $c + r$; that is,
$$\lim_{x \to (c+r)^-} f(x) = f(c+r).$$
2. If $\sum_{k=0}^{\infty} a_k (-r)^k$ converges, then f is continuous at $c - r$; that is,
$$\lim_{x \to (c-r)^+} f(x) = f(c-r).$$

Therefore, it is true that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$

9.10 Taylor and Maclaurin Series

Definition 9.106

If a function f has derivatives of all orders at $x = c$, then the series
$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$
is called the Taylor series for f at c. It is also called the Maclaurin series for f if $c = 0$.

Theorem 9.107: Convergence of Taylor Series

Let f be a function that has derivatives of all orders at $x = c$, and P_n be the n-th Taylor polynomial for f at c. If R_n, the remainder associated with P_n, has the property that
$$\lim_{n \to \infty} R_n(x) = 0 \qquad \forall\, x \in I$$
for some interval I, then the Taylor series for f converges and equals $f(x)$; that is,
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in I.$$

Corollary 9.108

Let f be a function that has derivatives of all orders in an open interval I containing c. If there exists $M > 0$ such that $\left|f^{(k)}(x)\right| \leqslant M$ for all $x \in I$ and each $k \in \mathbb{N}$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in I\,.$$

Proof. By the Taylor Theorem,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x)\,,$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

for some ξ between c and x. Since $\left|f^{(k)}(x)\right| \leqslant M$ for all $x \in I$ and $k \in \mathbb{N}$, we find that

$$\left|R_n(x)\right| \leqslant \frac{M}{(n+1)!}|x-c|^{n+1} \qquad \forall\, x \in I\,.$$

Therefore, by the fact that $\lim\limits_{n\to\infty} \dfrac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$ (the same reasoning as in Example 9.79), the Squeeze Theorem implies that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \forall\, x \in I$$

and Theorem 9.107 further shows that $f(x) = \sum\limits_{k=0}^{\infty} \dfrac{f^{(k)}(c)}{k!}(x-c)^k$. ◻

Example 9.109. Since the k-th derivatives of the sine function is bounded by 1; that is,

$$\left|\frac{d^k}{dx^k}\sin x\right| \leqslant 1 \qquad \forall\, x \in \mathbb{R} \text{ and } k \in \mathbb{N}\,,$$

Corollary 9.108 implies that for all $c \in \mathbb{R}$,

$$\sin x = \sum_{k=0}^{\infty} \frac{1}{k!}\sin\big(c + \frac{k\pi}{2}\big)(x-c)^k \qquad \forall\, x \in \mathbb{R}\,,$$

here we have used $\dfrac{d^k}{dx^k}\sin x = \sin\big(x + \dfrac{k\pi}{2}\big)$ to compute the k-th derivative of the sine function. In particular,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \forall\, x \in \mathbb{R}\,.$$

Similarly, for all $c \in \mathbb{R}$,

$$\cos x = \sum_{k=0}^{\infty} \frac{1}{k!} \cos\big(c + \frac{k\pi}{2}\big)(x-c)^k \qquad \forall\, x \in \mathbb{R}\,.$$

Example 9.110. Consider the natural exponential function $y = \exp(x)$. Note that for all real numbers $R > 0$, we have

$$\Big|\frac{d^k}{dx^k} e^x\Big| = e^x \leqslant e^R \qquad \forall\, x \in (-R, R) \text{ and } k \in \mathbb{N}\,;$$

thus Corollary 9.108 implies that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad \forall\, x \in (-R, R)\,.$$

Since the identity above holds for all $R > 0$, we conclude that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad \forall\, x \in \mathbb{R}\,.$$

Example 9.111 (Binomial Series)**.** In this example we consider the Maclaurin series, called the binomial series, of the function $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R}$ and $\alpha \neq \mathbb{N} \cup \{0\}$.

We compute the derivative of f and find that

$$\frac{d^k}{dx^k}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}\,.$$

Therefore,

$$f^{(k)}(0) = \frac{d^k}{dx^k}\Big|_{x=0}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$$

and the Maclaurin series for f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} x^k\,.$$

To see the radius of convergence of the Maclaurin series above, we use the ratio test and find that

$$\lim_{n\to\infty} \frac{\dfrac{\big|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n+1)+1)\big|}{(n+1)!}|x|^{n+1}}{\dfrac{\big|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\big|}{n!}|x|^n} = \lim_{n\to\infty} \frac{|\alpha-n|}{n+1}|x| = |x|\,;$$

thus the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k$ is 1. Moreover, by Taylor's theorem, for each $x \in (-1,1)$ there exists ξ between 0 and x such that

$$(1+x)^\alpha = \sum_{k=0}^{n} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k + R_n(x)\,,$$

where

$$R_n(x) = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{(n+1)!}(1+\xi)^{\alpha-n-1}x^{n+1}\,.$$

Similar to Example 9.76, we have

$$\big|R_n(x)\big| \leqslant \frac{\big|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)\big|}{(n+1)!}x^\alpha \qquad \forall\, x \in (0,1);$$

thus (without detail reasoning) we find that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \forall\, x \in (0,1)\,.$$

Therefore,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k \qquad \forall\, x \in (0,1)\,.$$

In fact,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k \qquad \forall\, x \in (-1,1)\,.$$

9.11 Exercise

Problem 9.1. Show that $\int_0^1 x^{-x}\,dx = \sum_{k=1}^{\infty} \frac{1}{k^k}$.

Hint: Write $x^{-x} = e^{-x\ln x}$ and use the Maclaurin series for exp to conclude that

$$\int_0^1 x^{-x}\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k(x\ln x)^k}{k!}\,dx\,.$$

Use the fact that $\int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k (x\ln x)^k}{k!}\,dx = \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k (x\ln x)^k}{k!}\,dx$. You will also need to recall the Gamma function.

Problem 9.2. Show that $\int_0^1 \frac{\ln x \ln(1+x)}{x}\,dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$.

Hint: Use (9.7.8) and rewrite the integral above as $\int_0^1 \ln x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{k-1}}{k}\,dx$. Assume that you know that

$$\int_0^1 \ln x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{k-1}}{k}\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^1 x^{k-1} \ln x\,dx\,.$$

Problem 9.3. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequence of real numbers such that the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge. Define $c_k = \sum_{j=0}^{k} a_j b_{k-j}$ and $C_n = \sum_{i=0}^{n} c_i$.

1. Show that if $\sum_{n=0}^{\infty} a_n$ converges absolutely, then

$$\lim_{n\to\infty} C_n = \Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big) \tag{9.11.1}$$

by completing the following.

(a) Show that $C_n = \sum_{k=0}^{n} a_{n-k} B_k$, where $B_k = \sum_{i=0}^{k} b_i$ is the k-th partial sum of the series $\sum_{i=0}^{\infty} b_i$.

(b) Let $A_k = \sum_{i=0}^{k} a_i$ be the k-th partial sum of the series $\sum_{i=0}^{\infty} a_i$, and $A = \lim_{n\to\infty} A_n$, $B = \sum_{n\to\infty} B_n$. Then

$$C_n - AB = \sum_{k=0}^{n} a_{n-k}(B_k - B) + (A_n - A)B\,.$$

Use the ε-N argument to show that $\lim_{n\to\infty} C_n = AB$.

2. $\sum_{n=0}^{\infty} c_n$ is called the Cauchy product of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Show that (9.11.1) may fail if both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges conditionally by looking at the example $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$ for all $n \in \mathbb{N}$.