# Calculus MA1002-B Final Exam

National Central University, Jun. 23, 2020

**Problem 1.** (15%) Let $a, b$ be positive constants. Evaluate the iterated integral

$$\int_0^a \Big( \int_0^b \exp\big( \max\{b^2x^2, a^2y^2\}\big)\, dy \Big) dx\,.$$

*Solution.* Let $R = [0,a] \times [0,b] = \big\{(x,y) \,\big|\, 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\big\}$, and

$$R_1 = \Big\{(x,y) \,\Big|\, 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant \frac{bx}{a}\Big\}, \qquad R_2 = \Big\{(x,y) \,\Big|\, 0 \leqslant x \leqslant b, 0 \leqslant x \leqslant \frac{ay}{b}\Big\}.$$

Then $R = R_1 \cup R_2$ and $R_1 \cap R_2$ has zero area. Therefore,

$$\begin{aligned}
\int_0^a \Big( \int_0^b \exp\big( \max\{b^2x^2, a^2y^2\}\big)\, dy \Big) dx &= \iint\limits_R \exp\big( \max\{b^2x^2, a^2y^2\}\big)\, dA \\
&= \iint\limits_{R_1} \exp\big( \max\{b^2x^2, a^2y^2\}\big)\, dA + \iint\limits_{R_2} \exp\big( \max\{b^2x^2, a^2y^2\}\big)\, dA \\
&= \iint\limits_{R_1} \exp(b^2x^2)\, dA + \iint\limits_{R_2} \exp(a^2y^2)\, dA \\
&= \int_0^a \Big( \int_0^{\frac{bx}{a}} \exp(b^2x^2)\, dy \Big) dx + \int_0^b \Big( \int_0^{\frac{ay}{b}} \exp(a^2y^2)\, dx \Big) dy \\
&= \int_0^a \frac{bx}{a} \exp(b^2x^2)\, dx + \int_0^b \frac{ay}{b} \exp(a^2y^2)\, dy \\
&= \frac{1}{2ab} \exp(b^2x^2)\Big|_{x=0}^{x=a} + \frac{1}{2ab} \exp(a^2y^2)\Big|_{y=0}^{y=b} = \frac{1}{ab}\big(e^{a^2b^2} - 1\big)\,.
\end{aligned}$$

□

**Problem 2.** (10%) Let $a, b$ be positive constants and $a < b$. Evaluate the integral $\displaystyle\int_0^1 \frac{x^b - x^a}{\ln x}\, dx$ by converting the integral into an iterated double integral and evaluating the iterated integral by changing the order of integration.

*Solution.* Treating $\dfrac{x^b - x^a}{\ln x}$ as $\dfrac{x^y}{\ln x}\Big|_{y=a}^{y=b}$, we find that

$$\begin{aligned}
\int_0^1 \frac{x^b - x^a}{\ln x}\, dx &= \int_0^1 \frac{x^y}{\ln x}\Big|_{y=a}^{y=b} dx = \int_0^1 \Big( \int_a^b \frac{\partial}{\partial y} \frac{x^y}{\ln x}\, dy \Big)\, dx = \int_0^1 \Big( \int_a^b x^y\, dy \Big) dx \\
&= \int_a^b \Big( \int_0^1 x^y\, dx \Big) dy = \int_a^b \frac{x^{y+1}}{y+1}\Big|_{x=0}^{x=1} dy = \int_a^b \frac{1}{y+1}\, dy \\
&= \ln(y+1)\Big|_{y=a}^{y=b} = \ln \frac{b+1}{a+1}\,.
\end{aligned}$$

□

**Problem 3.** (10%) Rewrite the iterated integral $\displaystyle\int_0^1 \Big[ \int_{x^2}^{\sqrt{x}} \Big( \int_{x^2}^{y} f(x,y,z)\, dz \Big) dy \Big] dx$ in the order $dx\, dy\, dz$.

*Solution.* We interchange the order of integration in the following order: $dzdydx \to dydzdx \to dydxdz \to dxdydz$ as follows:

$$\begin{aligned}\int_0^1 \Big[\int_{x^2}^{\sqrt{x}} \Big(\int_{x^2}^{y} f(x,y,z)\,dz\Big)dy\Big]dx &= \int_0^1 \Big[\int_{x^2}^{\sqrt{x}} \Big(\int_{z}^{\sqrt{x}} f(x,y,z)\,dy\Big)dz\Big]dx \\ &= \int_0^1 \Big[\int_{z^2}^{\sqrt{z}} \Big(\int_{z}^{\sqrt{x}} f(x,y,z)\,dy\Big)dx\Big]dz = \int_0^1 \Big[\int_{z}^{\sqrt[4]{z}} \Big(\int_{y^2}^{\sqrt{z}} f(x,y,z)\,dx\Big)dy\Big]dz\,.\end{aligned}$$

On the other hand, if we interchange the order of integration in the order $dzdydx \to dzdxdy \to dzdxdy \to dxdydz$, we obtain that

$$\begin{aligned}\int_0^1 \Big[\int_{x^2}^{\sqrt{x}} \Big(\int_{x^2}^{y} f(x,y,z)\,dz\Big)dy\Big]dx &= \int_0^1 \Big[\int_{y^2}^{\sqrt{y}} \Big(\int_{x^2}^{y} f(x,y,z)\,dz\Big)dx\Big]dy \\ &= \int_0^1 \Big[\int_{y^4}^{y} \Big(\int_{y^2}^{\sqrt{z}} f(x,y,z)\,dx\Big)dz\Big]dy = \int_0^1 \Big[\int_{z}^{\sqrt[4]{z}} \Big(\int_{y^2}^{\sqrt{z}} f(x,y,z)\,dx\Big)dy\Big]dz\,.\end{aligned}$$

□

**Problem 4.** Let $k > 0$ be a constant. Find the surface area of the cone $z = k\sqrt{x^2+y^2}$ that lies above the region $R = \big\{(x,y)\,\big|\, x^2+y^2 \leqslant 2y\big\}$ in the $xy$-plane by the following methods:

(1) (10%) Use the formula $\displaystyle\iint_R \sqrt{1+\|(\nabla f)(x,y)\|^2}\,dA$ directly.

(2) (15%) Find a parametrization of the cone above using $r,\theta$ (from the polar coordinate) as the parameters and make use of the formula $\displaystyle\iint_D \big\|(\boldsymbol{r}_r \times \boldsymbol{r}_\theta)(r,\theta)\big\|\,d(r,\theta)$.

(2) (15%) Find a parametrization of the cone above using $\rho,\theta$ (from the spherical coordinate) as the parameters and make use of the formula $\displaystyle\iint_D \big\|(\boldsymbol{r}_\rho \times \boldsymbol{r}_\theta)(\rho,\theta)\big\|\,d(\rho,\theta)$.

*Solution.* The region $R$ is the disk centered at $(0,1)$ with radius 1.

(1) Let $f(x,y) = k\sqrt{x^2+y^2}$. Then

$$f_x(x,y) = \frac{kx}{\sqrt{x^2+y^2}} \quad\text{and}\quad f_y(x,y) = \frac{ky}{\sqrt{x^2+y^2}}\,.$$

Therefore,

$$\sqrt{1+\|(\nabla f)(x,y)\|^2} = \sqrt{1+\frac{k^2x^2}{x^2+y^2}+\frac{k^2y^2}{x^2+y^2}} = \sqrt{k^2+1}$$

which implies that the desired surface area is given by

$$\iint_R \sqrt{k^2+1}\,dA = \sqrt{k^2+1} \times \text{the area of } R = \pi\sqrt{k^2+1}\,.$$

(2) Suppose that $(x,y,z)$ belongs to the surface. Then $z = k\sqrt{x^2+y^2}$. Using the spherical coordinate, $\rho\cos\phi = z = k\rho\sin\phi$; thus $\tan\phi = \dfrac{1}{k}$. This implies that $\sin\phi = \dfrac{1}{\sqrt{k^2+1}}$ and

$\cos\phi = \dfrac{k}{\sqrt{k^2+1}}$; thus a parametrization of the surface is given by

$$\boldsymbol{r}(\rho,\theta) = \Big(\frac{\rho\cos\theta}{\sqrt{1+k^2}}, \frac{\rho\sin\theta}{\sqrt{1+k^2}}, \frac{k\rho}{\sqrt{1+k^2}}\Big), \quad (\rho,\theta)\in D\,,$$

where, noting that $x^2+y^2 \leqslant 2y$ implies that $\rho^2\sin^2\phi \leqslant 2\rho\sin\theta\sin\phi$ in the spherical coordinate, we have

$$D = \Big\{(\rho,\theta)\,\Big|\, 0\leqslant\theta\leqslant\pi, \rho\leqslant 2\sqrt{k^2+1}\sin\theta\Big\}\,.$$

Therefore,

$$\boldsymbol{r}_\rho(\rho,\theta) = \Big(\frac{\cos\theta}{\sqrt{k^2+1}}, \frac{\sin\theta}{\sqrt{k^2+1}}, \frac{k}{\sqrt{k^2+1}}\Big) \quad\text{and}\quad \boldsymbol{r}_\theta(\rho,\theta) = \Big(-\frac{\rho\sin\theta}{\sqrt{k^2+1}}, \frac{\rho\cos\theta}{\sqrt{k^2+1}}, 0\Big)$$

so that

$$\|\boldsymbol{r}_\rho(\rho,\theta)\times\boldsymbol{r}_\theta(\rho,\theta)\| = \Big\|\Big(-\frac{k\rho\cos\theta}{k^2+1}, -\frac{k\rho\sin\theta}{k^2+1}, \frac{\rho}{k^2+1}\Big)\Big\| = \frac{\rho}{\sqrt{k^2+1}}\,.$$

Using the formula for parametric surface area, we find that the desired surface area is given by

$$\begin{aligned}\iint_D \frac{\rho}{\sqrt{k^2+1}}\,d(\rho,\theta) &= \int_0^\pi \Big(\int_0^{2\sqrt{k^2+1}\sin\theta} \frac{\rho}{\sqrt{k^2+1}}\,d\rho\Big)d\theta = \int_0^\pi \frac{\rho^2}{2\sqrt{k^2+1}}\Big|_{\rho=0}^{\rho=2\sqrt{k^2+1}\sin\theta} d\theta \\ &= 2\sqrt{k^2+1}\int_0^\pi \sin^2\theta\,d\theta = \sqrt{k^2+1}\int_0^\pi \big[1-\cos(2\theta)\big]\,d\theta = \pi\sqrt{k^2+1}\,.\end{aligned}$$ □

**Problem 5.** (15%) Evaluate the double integral $\displaystyle\iint_R f(x,y)\,dA$, where $f(x,y) = e^{-(x^2+xy+y^2)}$ and $R$ is the region $R = \big\{(x,y)\,\big|\,x^2+xy+y^2\leqslant 1\big\}$ (which is an ellipse).

*Solution.* Note that $x^2+xy+y^2 = \big(x+\frac{y}{2}\big)^2 + \frac{3y^2}{4}$; thus we make a change of variables $u = x+\frac{y}{2}$ and $v = \frac{\sqrt{3}y}{2}$ so that the corresponding region of $R$ in the $uv$-plane is

$$R' = \big\{(u,v)\,\big|\,u^2+v^2\leqslant 1\big\}\,.$$

Moreover,

$$\frac{\partial(x,y)}{\partial(u,v)} = \Big(\frac{\partial(u,v)}{\partial(x,y)}\Big)^{-1} = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2}\end{vmatrix}^{-1} = \frac{2}{\sqrt{3}}\,.$$

Therefore, using the polar coordinate and the change of variables formula,

$$\begin{aligned}\iint_R f(x,y)\,dA &= \frac{2}{\sqrt{3}}\iint_{u^2+v^2\leqslant 1} e^{-(u^2+v^2)}\,d(u,v) = \frac{2}{\sqrt{3}}\int_0^{2\pi}\Big(\int_0^1 e^{-r^2}r\,dr\Big)d\theta = -\frac{2\pi}{\sqrt{3}}e^{-r^2}\Big|_{r=0}^{r=1} \\ &= \frac{2\pi(1-e^{-1})}{\sqrt{3}}\,.\end{aligned}$$ □

**Problem 6.** For a vector $\boldsymbol{v} = (v_1,v_2,v_3)\in\mathbb{R}^3$, let $\|\boldsymbol{v}\| = \sqrt{v_1^2+v_2^2+v_3^2}$ denote the length of $\boldsymbol{v}$. Suppose that $\boldsymbol{c}\in\mathbb{R}^3$ is a unit vector; that is, $\|\boldsymbol{c}\| = 1$. Show that

$$\iiint_{B(\boldsymbol{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\boldsymbol{c}\cdot\boldsymbol{x})\,dV = 2\pi r - \pi\sin(2r)\,,$$

where $B(\boldsymbol{0},r)$ is the ball centered at the origin with radius $r$, via the following steps.

(1) (10%) Let O be an orthogonal $3 \times 3$ matrix (that is, $\mathrm{O}^{\mathrm{T}}\mathrm{O} = \mathrm{O}\mathrm{O}^{\mathrm{T}} = \mathrm{I}_{3\times 3}$). Show that the change of variable $\boldsymbol{x} = \mathrm{O}\boldsymbol{y}$ implies that

$$\iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\boldsymbol{c}\cdot\boldsymbol{x})\,dV = \iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\mathrm{O}^{\mathrm{T}}\boldsymbol{c}\cdot\boldsymbol{x})\,dV\,.$$

**Hint**: Compute the Jacobian $\dfrac{\partial(x_1,x_2,x_3)}{\partial(y_1,y_2,y_3)}$.

(2) (5%) Use the fact that there exists an orthogonal $3 \times 3$ matrix O such that $\mathrm{O}^{\mathrm{T}}\boldsymbol{c} = (0,0,1)^{\mathrm{T}}$ to conclude that

$$\iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\boldsymbol{c}\cdot\boldsymbol{x})\,dV = \iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos x_3\,dV\,. \qquad (\star)$$

(3) (10%) Use the spherical coordinates to computer the triple integral on the right-hand side of $(\star)$ (in the order $d\theta\,d\phi\,d\rho$) and obtain the desired result.

*Proof.* (1) Suppose that O be an orthogonal $3 \times 3$ matrix (so that $\mathrm{O}^{\mathrm{T}}\mathrm{O} = \mathrm{O}\mathrm{O}^{\mathrm{T}} = \mathrm{I}_3$). Then

(a) The corresponding region of $B(\mathbf{0},r)$ in the $y$-space is still the ball centered at the origin with radius $r$ since $(\mathrm{O}\boldsymbol{u})\cdot(\mathrm{O}\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{v}$ for all $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^3$.

(b) The Jacobin of $\boldsymbol{x}$ w.r.t. $\boldsymbol{y}$ is 1 or $-1$ since

$$\frac{\partial(x_1,x_2,x_3)}{\partial(y_1,y_2,y_3)} = \det(\mathrm{O})$$

and $\det(\mathrm{O})^2 = \det(\mathrm{O})\det(\mathrm{O}^{\mathrm{T}}) = \det(\mathrm{I}_{3\times 3}) = 1$.

Therefore, the change of variable $\boldsymbol{x} = \mathrm{O}\boldsymbol{y}$ implies that

$$\begin{aligned}
&\iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\boldsymbol{c}\cdot\boldsymbol{x})\,dV \\
&\quad= \iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\mathrm{O}\boldsymbol{y}\|}{\|\mathrm{O}\boldsymbol{y}\|}\cos(\boldsymbol{c}\cdot\mathrm{O}\boldsymbol{y})\big|\det(\mathrm{O})\big|\,dV = \iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|}\cos(\mathrm{O}^{\mathrm{T}}\boldsymbol{c}\cdot\boldsymbol{y})\,d(y_1,y_2,y_3) \\
&\quad= \iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\mathrm{O}^{\mathrm{T}}\boldsymbol{c}\cdot\boldsymbol{x})\,dV\,,
\end{aligned}$$

where the last equality follows from that $\boldsymbol{y}$ is an dummy variable.

(2) Choose an orthogonal matrix O such that $\mathrm{O}^{\mathrm{T}}\boldsymbol{c} = (0,0,1)$ (there is always such an orthogonal matrix since O). Then the above identity shows that

$$\iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos(\boldsymbol{c}\cdot\boldsymbol{x})\,dV = \iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos((0,0,1)\cdot\boldsymbol{x})\,dV = \iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos x_3\,dV$$

which concludes $(\star)$.

(3) Using the spherical coordinates,

$$\iiint\limits_{B(\mathbf{0},r)} \frac{\sin\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\cos x_3\,dV = \int_0^r \Big[\int_0^\pi \Big(\int_0^{2\pi} \frac{\sin\rho}{\rho}\cos(\rho\cos\phi)\rho^2\sin\phi\,d\theta\Big)d\phi\Big]d\rho$$
$$= 2\pi\int_0^r\Big(\int_0^\pi \sin\rho\cos(\rho\cos\phi)\rho\sin\phi\,d\phi\Big)d\rho = -2\pi\int_0^r\Big(\int_0^\pi \frac{\partial}{\partial\phi}\big[\sin\rho\sin(\rho\cos\phi)\big]d\phi\Big)d\rho$$
$$= -2\pi\int_0^r \big[\sin\rho\sin(\rho\cos\phi)\big]\Big|_{\phi=0}^{\phi=\pi}d\rho = 4\pi\int_0^r \sin^2\rho\,d\rho = 2\pi\int_0^r \big[1-\cos(2\rho)\big]\,d\rho$$
$$= 2\pi\big[\rho - \frac{1}{2}\sin(2\rho)\big]\Big|_{\rho=0}^{\rho=r} = 2\pi r - \pi\sin(2r)\,.$$

□