Calculus MA1002-B Quiz 12

National Central University, Jun. 16 2020

學號：____________________ 姓名：____________________

Problem 1. (4pts) Evaluate the double integral $\int_0^2 \Big(\int_{y^2}^4 \sqrt{x}\sec^2 x\,dx\Big)dy$.

Solution. The domain of integration R is $\big\{(x,y)\,\big|\,0\leqslant y\leqslant 2, y^2\leqslant x\leqslant 4\big\}$ which can be rewritten as

$$R=\big\{(x,y)\,\big|\,0\leqslant x\leqslant 4, 0\leqslant y\leqslant \sqrt{x}\big\}\,.$$

Therefore, the Fubini Theorem implies that

$$\int_0^2 \Big(\int_{y^2}^4 \sqrt{x}\sec^2 x\,dx\Big)dy = \iint_R \sqrt{x}\sec^2 x\,dA = \int_0^4 \Big(\int_0^{\sqrt{x}} \sqrt{x}\sec^2 x\,dy\Big)dx = \int_0^4 x\sec^2 x\,dx\,.$$

Let $u=x$ and $v=\tan x$ (so that $dv=\sec^2 x\,dx$), integrating by parts we have

$$\int_0^4 x\sec^2 x\,dx = x\tan x\Big|_{x=0}^{x=4} - \int_0^4 \tan x\,dx = 4\tan 4 - \ln|\sec x|\Big|_{x=0}^{x=4} = 4\tan 4 - \ln|\sec 4|\,;$$

thus $\int_0^2 \Big(\int_{y^2}^4 \sqrt{x}\sec^2 x\,dx\Big)dy = 4\tan 4 - \ln|\sec 4|$. □

Problem 2. (3pts) 假設地球半徑是 R 公里，求在北緯 30 度到 45 度，東經 120 度到 135 度這個區域的面積是多少平方公里？請先寫出以經度 θ（東經為正西經為負，範圍 $-\pi$ 到 π）及緯度 ϕ（北緯為正南緯為負，範圍 $-\frac{\pi}{2}$ 到 $\frac{\pi}{2}$）的一組地球表面參數式之後，以此參數式求所求曲面面積。

Solution. Let θ,ϕ denote the latitude (經度) and longitude (緯度) (in radian) of the earth. Then

$$\boldsymbol{r}(\theta,\phi)=(R\cos\theta\cos\phi, R\sin\theta\cos\phi, R\sin\phi)\,,\qquad \theta\in[-\pi,\pi], \phi\in\Big[-\frac{\pi}{2},\frac{\pi}{2}\Big]$$

is a parametrization of the surface of the earth. Moreover

$$\boldsymbol{r}_\theta(\theta,\phi)=(-R\sin\theta\cos\phi, R\cos\theta\cos\phi, 0) \quad\text{and}\quad \boldsymbol{r}_\phi(\theta,\phi)=(-R\cos\theta\sin\phi, -R\sin\theta\sin\phi, R\cos\phi)$$

which implies that

$$\big\|\boldsymbol{r}_\theta(\theta,\phi)\times\boldsymbol{r}_\phi(\theta,\phi)\big\| = \big\|(R^2\cos\theta\cos^2\phi, R^2\sin\theta\cos^2\phi, R^2\cos\phi)\big\| = R^2\cos\phi\,.$$

Therefore, the desired surface area is given by

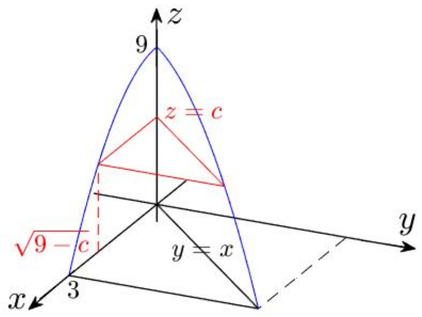
$$\iint_{[\frac{2\pi}{3},\frac{3\pi}{4}]\times[\frac{\pi}{6},\frac{\pi}{4}]} \big\|\boldsymbol{r}_\theta(\theta,\phi)\times\boldsymbol{r}_\phi(\theta,\phi)\big\|\,d(\theta,\phi) = R^2\Big(\frac{3\pi}{4}-\frac{2\pi}{3}\Big)\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos\phi\,d\phi = \frac{\pi R^2(\sqrt{2}-1)}{24}\,.$$ □

（背面尚有題目）

Problem 3. (3pts) Rewrite the iterated integral $\int_0^3 \Big[\int_0^x \Big(\int_0^{9-x^2} f(x,y,z)\,dz\Big)dy\Big]dx$ in the order $dxdydz$.

Solution. We present two methods.

Method 1: This method involves graphing the solid region of integration. The domain of integration is the solid region above the triangular region $R = \big\{(x,y)\,\big|\,0 \leqslant y \leqslant 3, y \leqslant x \leqslant 3\big\}$ and beneath the graph $z = 9 - x^2$; thus the domain of integration is



Therefore, for fixed $z = c$, the domain of integration (the red triangle) is $\big\{(x,y)\,\big|\,0 \leqslant y \leqslant \sqrt{9-c}, y \leqslant x \leqslant \sqrt{9-c}\big\}$; thus

$$\int_0^3 \Big[\int_0^x \Big(\int_0^{9-x^2} f(x,y,z)\,dz\Big)dy\Big]dx = \int_0^9 \Big[\int_0^{\sqrt{9-z}} \Big(\int_y^{\sqrt{9-z}} f(x,y,z)\,dx\Big)dy\Big]dz\,.$$

Method 2: This method only involves graphics the planar region of integration, and is to interchange the order of integration of two variables each time (but several times). Since we would like to obtain integration in the order $dxdydz$ but we start from $dzdydx$, we can first interchange the order of integration to $dydzdx$ and then $dydxdz$ and finally $dxdydz$. Using this idea, we have

$$\int_0^3 \Big[\int_0^x \Big(\int_0^{9-x^2} f(x,y,z)\,dz\Big)dy\Big]dx = \int_0^3 \Big[\int_y^3 \Big(\int_0^{9-x^2} f(x,y,z)\,dz\Big)dx\Big]dy$$

then

$$\int_0^3 \Big[\int_y^3 \Big(\int_0^{9-x^2} f(x,y,z)\,dz\Big)dx\Big]dy = \int_0^3 \Big[\int_0^{9-y^2} \Big(\int_y^{\sqrt{9-z}} f(x,y,z)\,dx\Big)dz\Big]dy$$

and finally

$$\int_0^3 \Big[\int_0^{9-y^2} \Big(\int_y^{\sqrt{9-z}} f(x,y,z)\,dx\Big)dz\Big]dy = \int_0^9 \Big[\int_0^{\sqrt{9-z}} \Big(\int_y^{\sqrt{9-z}} f(x,y,z)\,dx\Big)dy\Big]dz\,.$$

One can also interchange in the following way:

$$\int_0^3 \Big[\int_0^x \Big(\int_0^{9-x^2} f(x,y,z)\,dz\Big)dy\Big]dx = \int_0^3 \Big[\int_0^{9-x^2} \Big(\int_0^x f(x,y,z)\,dy\Big)dz\Big]dx$$

$$= \int_0^9 \Big[\int_0^{\sqrt{9-z}} \Big(\int_0^x f(x,y,z)\,dy\Big)dx\Big]dz = \int_0^9 \Big[\int_0^{\sqrt{9-z}} \Big(\int_y^{\sqrt{9-z}} f(x,y,z)\,dx\Big)dy\Big]dz\,. \qquad \square$$