Calculus MA1002-B Quiz 11

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Problem 1. (5pts) Use the method of Lagrange multipliers to find the extrema of $f(x, y, z) = xy^2z$ subject to the constraint $x^2 + y^2 + z^2 = 4$.

Solution. Let $g(x, y, z) = x^2 + y^2 + z^2 - 4$. Then $(\nabla g)(x, y, z) \neq \mathbf{0}$ if $g(x, y, z) = 0$; thus if f, subject to the constraint $g = 0$, attains its extrema at (x_0, y_0, z_0), there exists $\lambda \in \mathbb{R}$ such that

$$(y_0^2 z_0, 2x_0y_0z_0, x_0y_0^2) = (\nabla f)(x_0, y_0, z_0) = \lambda(\nabla g)(x_0, y_0, z_0) = 2\lambda(x_0, y_0, z_0)\,.$$

Since $g(x_0, y_0, z_0) = 0$, we find that

$$(x_0, y_0, z_0) \cdot (y_0^2 z_0, 2x_0y_0z_0, x_0y_0^2) = 2\lambda(x_0^2 + y_0^2 + z_0^2) = 8\lambda$$

which implies that $x_0y_0^2z_0 = 2\lambda$. Therefore, $(y_0^2z_0, 2x_0y_0z_0, x_0y_0^2) = x_0y_0^2z_0(x_0, y_0, z_0)$ which shows that

$$(x_0^2 - 1)y_0^2 z_0 = x_0y_0z_0(y_0^2 - 2) = x_0y_0^2(z_0^2 - 1) = 0\,.$$

Together with $x_0^2 + y_0^2 + z_0^2 = 4$, we find that (x_0, y_0, z_0) can be

$$(\pm 2, 0, 0), (0, \pm 2, 0), (0, 0, \pm 2), (\pm 1, \pm\sqrt{2}, \pm 1).$$

This implies that f, subject to $g = 0$, attains its maximum at $(1, \sqrt{2}, 1)$ with value $f(1, \sqrt{2}, 1) = 2$.
□

Problem 2. (5pts) Use the method of Lagrange multipliers to find the maximum of $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 - z^2 = 0$ and $x + 2z = 4$.

Solution. Let $g(x, y, z) = x^2 + y^2 - z^2$ and $h(x, y, z) = x + 2z - 4$. Then

$$(\nabla g)(x, y, z) \times (\nabla h)(x, y, z) = (2x, 2y, -2z) \times (1, 0, 2) = (4y, -4x - 2z, -2y)$$

which, together with $g(x, y, z) = h(x, y, z) = 0$, is never zero. Therefore, if f, subject to the constraints $g = h = 0$, attains its maximum at (x_0, y_0, z_0), then there exist $\lambda, \mu \in \mathbb{R}$ such that

$$(0, 0, 1) = (\nabla f)(x_0, y_0, z_0) = \lambda(\nabla g)(x_0, y_0, z_0) + \mu(\nabla h)(x_0, y_0, z_0) = \lambda(2x_0, 2y_0, -2z_0) + \mu(1, 0, 2)\,.$$

Therefore, $\lambda \neq 0$ and $(x_0, y_0, z_0, \lambda, \mu)$ satisfies that

$$2\lambda x_0 + \mu = 0 \tag{0.1a}$$
$$2\lambda y_0 = 0 \tag{0.1b}$$
$$-2\lambda z_0 + 2\mu = 1 \tag{0.1c}$$
$$x_0^2 + y_0^2 - z_0^2 = 0 \tag{0.1d}$$
$$x_0 + 2z_0 = 4\,. \tag{0.1e}$$

Therefore, (0.1b) shows that $y_0 = 0$ which, using (0.1d), further implies that $x_0 = \pm z_0$; thus (0.1e) leads to that $x_0 = z_0 = \dfrac{4}{3}$ or $x_0 = -z_0 = -4$. Therefore, f, subject to $g = h = 0$, attains its maximum at $(-4, 0, 4)$ with value $f(-4, 0, 4) = 4$. □