Calculus MA1002-B Quiz 10

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Problem 1. (5pts) Use the second partials test to determine the relative extrema and saddle points of the function $f(x,y) = (x^2+y^2)e^{-x}$.

Solution. First we look for critical points (x,y) satisfying

$$f_x(x,y) = 2xe^{-x} - (x^2+y^2)e^{-x} = 0 \quad \text{and} \quad f_y(x,y) = 2ye^{-x}\,.$$

Therefore, there are only two critical points $(0,0)$ and $(2,0)$. Note that

$$\begin{aligned} f_{xx}(x,y) &= 2e^{-x} - 4xe^{-x} + (x^2+y^2)e^{-x}\,, \\ f_{xy}(x,y) &= -2ye^{-x}\,, \quad f_{yy}(x,y) = 2e^{-x}\,. \end{aligned}$$

1. At the point $(0,0)$, $f_{xx}(0,0) = 2$ and $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 4 > 0$; thus the second partials test implies that $f(0,0)$ is a relative minimum of f.
2. At the point $(2,0)$, $f_{xx}(2,0) = -2e^{-2}$ and $f_{xx}(2,0)f_{yy}(2,0) - f_{xy}(2,0)^2 = -4e^{-4} < 0$; thus the second partials test implies that $(2,0,f(2,0))$ is a saddle point of f. □

Problem 2. (5pts) Find the absolute extrema of the function $f(x,y) = 3x^2 + 2y^2 - 4y$ over the region R in the xy-plane bounded by the graphs of $y = x^2$ and $y = 4$.

Solution. First we look for local extrema in the interior of the region. In this case, we look for the critical points of f that satisfy

$$f_x(x,y) = 6x = 0 \quad \text{and} \quad f_y(x,y) = 4y - 4 = 0\,.$$

Therefore, there is only one critical point $(0,1)$ (which is inside R) of f and $f(0,1) = -2$.

Next we look for the extrema on the boundary of R. On the lower part of the boundary $y = x^2$,

$$f(x,y) = f(x,x^2) = 3x^2 + 2x^4 - 4x^2 = 2x^4 - x^2$$

which, on the interval $[-2,2]$, has critical points at $x = 0$ and $x = \pm\frac{1}{2}$. Therefore, $f(x,x^2)$ attains its maximum at $x = \pm 2$ (with value $f(\pm 2, 4) = 28$) and attains its minimum at $x = \pm\frac{1}{2}$ (with value $f\big(\pm\frac{1}{2}, \frac{1}{4}\big) = -\frac{1}{8}\big)$. On the upper part of the boundary $y = 4$,

$$f(x,y) = f(x,4) = 3x^2 + 16$$

which, on the interval $[-2,2]$, attains its maximum at $x = \pm 2$ (with value $f(\pm 2,4) = 28$) and attains its minimum at $x = 0$ (with value $f(0,0) = 0$).

Comparing the values computed above, we find that the absolute maximum of f over R occurs at $(\pm 2, 4)$ with value $f(\pm 2, 4) = 28$ and the absolute minimum of f over R occurs at $(0,1)$ with value $f(0,1) = -2$. □