Calculus MA1002-B Quiz 08

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Problem 1. Let U be an open set in $\mathbb{R}^2$, $f : U \to \mathbb{R}$ be a real-valued function of two variables, and $(a,b) \in U$.

1. (2pts) State the definition of that f is differentiable at (a,b).
2. (3pts) Show that if f is differentiable at (a,b), then f is continuous at (a,b).

Solution. 1. f is said to be differentiable at (a,b) if there exist $A, B \in \mathbb{R}$ such that

$$\lim_{(x,y)\to(a,b)} \frac{\big|f(x,y) - f(a,b) - A(x-a) - B(y-b)\big|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0\,.$$

2. Suppose that f is differentiable at (a,b). By the definition of differentiability,

$$\lim_{(x,y)\to(a,b)} \frac{\big|f(x,y) - f(a,b) - A(x-a) - B(y-b)\big|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0\,.$$

Then by the fact that $\lim\limits_{(x,y)\to(a,b)} A(x-a) = \lim\limits_{(x,y)\to(a,b)} B(y-b) = 0$, we find that

$$\begin{aligned}
\lim_{(x,y)\to(a,b)} \big|f(x,y) - f(a,b)\big| &= \lim_{(x,y)\to(a,b)} \big|f(x,y) - f(a,b) - A(x-a) - B(y-b)\big| \\
&= \lim_{(x,y)\to(a,b)} \Big(\frac{\big|f(x,y) - f(a,b) - A(x-a) - B(y-b)\big|}{\sqrt{(x-a)^2 + (y-b)^2}}\sqrt{(x-a)^2 + (y-b)^2}\Big) \\
&= \lim_{(x,y)\to(a,b)} \frac{\big|f(x,y) - f(a,b) - A(x-a) - B(y-b)\big|}{\sqrt{(x-a)^2 + (y-b)^2}} \cdot \lim_{(x,y)\to(a,b)} \sqrt{(x-a)^2 + (y-b)^2} = 0\,.
\end{aligned}$$

Therefore, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. □

Problem 2. (5pts) Show that the function $f(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,, \end{cases}$ is not differentiable at $(0,0)$ but $f_x(0,0)$ and $f_y(0,0)$ both exist.

Proof. First we compute $f_x(0,0)$ and $f_y(0,0)$. By definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \quad \text{and} \quad f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0\,.$$

Therefore, $f_x(0,0) = f_y(0,0) = 0$ both exist. However, f is not differentiable at $(0,0)$ since the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{\big|f(x,y) - f(0,0) - f_x(0,0)(x-0) - f_y(0,0)(y-0)\big|}{\sqrt{x^2+y^2}}$ D.N.E.: when (x,y) approaches $(0,0)$ along the line $y = mx$, then

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{\big|f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y\big|}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{2|m|x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{2|m|}{1+m^2} = \frac{2|m|}{1+m^2}$$

which has different values for different m. □