Calculus MA1002-B Quiz 07

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Problem 1. (3pts) Determine whether the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$ exists or not. Explain your answer.

Solution. Let $m \in \mathbb{R}$. Along the curve $y = mx^{\frac{3}{2}}$, $x > 0$, we find that

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx^{3/2},x>0}} \frac{x^3y^2}{x^6+y^4} = \lim_{x\to0^+} \frac{m^2x^6}{x^6+m^4x^6} = \lim_{x\to0^+} \frac{m^2}{1+m^4}$$

which varies if m varies. Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$ D.N.E. □

Problem 2. (3pts) Find the value of $\dfrac{\partial x}{\partial y}$ at the point $(1,-1,-3)$ if the equation

$$xz + y\ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Proof. Assume that $x = f(y,z)$ for some function f near the point $(1,-1,-3)$. Then

$$zf(y,z) + y\ln f(y,z) - f(y,z)^2 + 4 = 0\,.$$

Taking the partial derivative of the equation above w.r.t. y, we find that

$$zf_y(y,z) + \ln f(y,z) + \frac{yf_y(y,z)}{f(y,z)} - 2f(y,z)f_y(y,z) = 0\,;$$

thus

$$-3f_y(-1,-3) + \ln f(-1,-3) + \frac{-f_y(-1,-3)}{f(-1,-3)} - 2f(-1,-3)f_y(-1,-3) = 0\,.$$

Since $f(-1,-3) = 1$, we have

$$\frac{\partial x}{\partial y}\Big|_{(x,y,z)=(1,-1,-3)} = f_y(-1,-3) = 0\,.$$

□

Problem 3. (4pts) Let $f(x,y) = \displaystyle\int_1^y \sqrt{x^4+t^4}\,dt$. Show that

$$xf_x(x,y) + yf_y(x,y) = 3f(x,y) + \sqrt{1+x^4} \qquad \forall\, x>0, y\in\mathbb{R}\,.$$

Proof. Let $x > 0$ and $y \in \mathbb{R}$. By the substitution of variable $t = xs$,

$$f(x,y) = \int_1^y \sqrt{x^4+t^4}\,dt = \int_{\frac{1}{x}}^{\frac{y}{x}} x\sqrt{x^4+x^4s^4}\,ds = x^3\int_{\frac{1}{x}}^{\frac{y}{x}} \sqrt{1+s^4}\,ds\,.$$

By the product rule and the Fundamental Theorem of Calculus,

$$\begin{aligned} f_x(x,y) &= 3x^2\int_{\frac{1}{x}}^{\frac{y}{x}} \sqrt{1+s^4}\,ds + x^3\Big[\sqrt{1+y^4x^{-4}}\cdot\frac{\partial}{\partial x}\frac{y}{x} - \sqrt{1+x^{-4}}\cdot\frac{\partial}{\partial x}\frac{1}{x}\Big] \\ &= \frac{3f(x,y)}{x} - \frac{y\sqrt{x^4+y^4}}{x} + \frac{\sqrt{1+x^4}}{x}\,; \end{aligned}$$

thus by the fact that $f_y(x,y) = \sqrt{x^4+y^4}$, we find that

$$xf_x(x,y) + yf_y(x,y) = 3f(x,y) + \sqrt{1+x^4}\,.$$

□