Calculus MA1002-B Quiz 05

National Central University, Apr. 21 2020

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Problem 1. (5pts) Show that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$ for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in space.

Proof. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, and $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$. Then

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\,,\\ \mathbf{c}\times\mathbf{d} &= (c_2d_3 - c_3d_2)\mathbf{i} + (c_3d_1 - c_1d_3)\mathbf{j} + (c_1d_2 - c_2d_1)\mathbf{k}\,;\end{aligned}$$

thus

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (a_2b_3 - a_3b_2)(c_2d_3 - c_3d_2) + (a_3b_1 - a_1b_3)(c_3d_1 - c_1d_3) + (a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)\\ &= a_2c_2b_3d_3 + a_3c_3b_2d_2 - a_3d_3b_2c_2 - a_2d_2b_3c_3 + a_3c_3b_1d_1 + a_1c_1b_3d_3\\ &\quad -a_1d_1b_3c_3 - a_3d_3b_1c_1 + a_1c_1b_2d_2 + a_2c_2b_1d_1 - a_2d_2b_1c_1 - a_1d_1b_2c_2\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix} &= (a_1c_1 + a_2c_2 + a_3c_3)(b_1d_1 + b_2d_2 + b_3d_3) - (a_1d_1 + a_2d_2 + a_3d_3)(b_1c_1 + b_2c_2 + b_3c_3)\\ &= a_1c_1b_2d_2 + a_1c_1b_3d_3 + a_2c_2b_1d_1 + a_2c_2b_3d_3 + a_3c_3b_1d_1 + a_3c_3b_2d_2\\ &\quad - a_1d_1b_2c_2 - a_1d_1b_3c_3 - a_2d_2b_1c_1 - a_2d_2b_3c_3 - a_3d_3b_1c_1 - a_3d_3b_2c_2\end{aligned}$$

Compare the two identities term by term, we find that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$. □

Problem 2. (3pts) Let four points P, Q, R, S, given in Cartesian coordinate, be $P = (2, 1, 0)$, $Q = (1, 0, -1)$, $R = (2, 2, -1)$ and $S = (1, 2, -2)$. Find the distance of P to the plane passing through the points Q, R, S using cross product.

Solution. Give P, Q, R, S above, we have $\overrightarrow{QP} = (1, 1, 1)$, $\overrightarrow{QR} = (1, 2, 0)$ and $\overrightarrow{QS} = (0, 2, -1)$. The volume of a parallelepiped with vectors $\overrightarrow{QP}$, $\overrightarrow{QR}$, and $\overrightarrow{QS}$ as adjacent edges is

$$\left|\overrightarrow{QP}\cdot\big(\overrightarrow{QR}\times\overrightarrow{QS}\big)\right| = \left|\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & -1 \end{vmatrix}\right| = \big|-2 + 2 - (-1)\big| = 1\,.$$

The desired distance is then the volume above divided by the area of base parallelogram spanned by $\overrightarrow{QR}$ and $\overrightarrow{QS}$: $\dfrac{1}{\|\overrightarrow{QR}\times\overrightarrow{QS}\|} = \dfrac{1}{\|(-2, 1, 2)\|} = \dfrac{1}{3}$. □

Problem 3. (2pts) Replace the polar equations $r\sin\theta - \ln r = \ln\cos\theta$ with equivalent Cartesian equations.

Solution. Let (x, y) be the Cartesian coordinate. Then $x = r\cos\theta$ and $y = r\sin\theta$. Since (r, θ) satisfies the polar equation $r\sin\theta - \ln r = \ln\cos\theta)$, (x, y) must satisfies

$$y = r\sin\theta = \ln r + \ln\cos\theta = \ln(r\cos\theta) = \ln x\,.$$ □