Exercise Problem Sets 12

May. 29. 2020

Problem 1. Use the method of Lagrange multipliers to complete the following.

(1) Maximize $f(x,y) = \sqrt{6 - x^2 - y^2}$ subject to the constraint $x + y - 2 = 0$.

(2) Minimize $f(x,y) = 3x^2 - y^2$ subject to the constraint $2x - 2y + 5 = 0$.

(3) Minimize $f(x,y) = x^2 + y^2$ subject to the constraint $xy^2 = 54$.

(4) Maximize $f(x,y,z) = e^{xyz}$ subject to the constraint $2x^2 + y^2 + z^2 = 24$.

(5) Maximize $f(x,y,z) = \ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1)$ subject to the constraint $x^2+y^2+z^2 = 12$.

(6) Maximize $f(x,y,z) = x + y + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

(7) Maximize $f(x,y,z,t) = x + y + z + t$ subject to the constraint $x^2 + y^2 + z^2 + t^2 = 1$.

(8) Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2z = 6$ and $x + y = 12$.

(9) Maximize $f(x,y,z) = z$ subject to the constraints $x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

(10) Maximize $f(x,y,z) = yz + xy$ subject to the constraint $xy = 1$ and $y^2 + z^2 = 1$.

Problem 2. Use the method of Lagrange multipliers to find the extreme values of the function $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$ subject to the constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$.

Problem 3. (1) Use the method of Lagrange multipliers to show that the product of three positive numbers x, y, and z, whose sum has the constant value S, is a maximum when the three numbers are equal. Use this result to show that

$$\frac{x+y+z}{3} \geqslant \sqrt[3]{xyz} \qquad \forall\, x, y, z > 0\,.$$

(2) Generalize the result of part (1) to prove that the product $x_1 x_2 x_3 \cdots x_n$ is maximized, under the constraint that $\sum_{i=1}^{n} x_i = S$ and $x_i \geqslant 0$ for all $1 \leqslant i \leqslant n$, when

$$x_1 = x_2 = x_3 = \cdots = x_n\,.$$

Then prove that

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n} \qquad \forall\, x_1, x_2, \cdots, x_n \geqslant 0\,.$$

Problem 4. (1) Maximize $\sum_{i=1}^{n} x_i y_i$ subject to the constraints $\sum_{i=1}^{n} x_i^2 = 1$ and $\sum_{i=1}^{n} y_i^2 = 1$.

(2) Put $x_i = \dfrac{a_i}{\sqrt{\sum\limits_{j=1}^{n} a_j^2}}$ and $y_i = \dfrac{b_i}{\sqrt{\sum\limits_{j=1}^{n} b_j^2}}$ to show that

$$\sum_{i=1}^{n} a_i b_i \leqslant \sqrt{\sum_{j=1}^{n} a_j^2}\sqrt{\sum_{j=1}^{n} b_j^2}$$

for any numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

Problem 5. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

Problem 6. If the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of the ellipse?

Problem 7. (1) Use the method of Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

(2) Use the method of Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.

Hint: Use Heron's formula for the area:

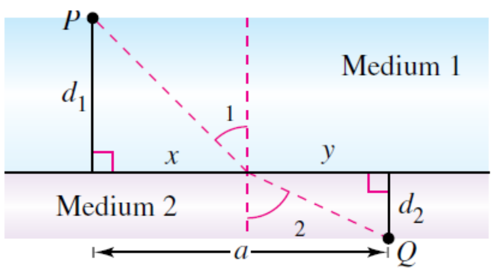
$$A = \sqrt{s(s-x)(s-y)(s-z)},$$

where $s = \dfrac{p}{2}$ and x, y, z are the lengths of the sides.

Problem 8. When light waves traveling in a transparent medium strike the surface of a second transparent medium, they tend to "bend" in order to follow the path of minimum time. This tendency is called refraction and is described by Snell' s Law of Refraction,

$$\frac{\sin\theta_1}{\mathrm{v}_1} = \frac{\sin\theta_2}{\mathrm{v}_2},$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure, and v_1 and v_2 are the velocities of light in the two media. Use the method of Lagrange multipliers to derive this law using $x + y = a$.



Problem 9. A set $C \subseteq \mathbb{R}^n$ is said to be convex if

$$t\boldsymbol{x} + (1-t)\boldsymbol{y} \in C \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in C \text{ and } t \in [0,1]\,.$$

(一個 $\mathbb{R}^n$ 中的集合 C 被稱為凸集合如果 C 中任兩點 $\boldsymbol{x}$ 與 $\boldsymbol{y}$ 之連線所形成的線段也在 C 中)。

Suppose that $C \subseteq \mathbb{R}^n$ is a convex set, and $f : C \to \mathbb{R}$ be a differentiable real-valued function. Show that if f on C attains its minimum at a point $\boldsymbol{x}^*$, then

$$(\nabla f)(\boldsymbol{x}^*) \cdot (\boldsymbol{x} - \boldsymbol{x}^*) \geqslant 0 \qquad \forall\, \boldsymbol{x} \in C\,. \tag{$\star$}$$

Hint: Recall that $(\nabla f)(\boldsymbol{x}^*) \cdot (\boldsymbol{x} - \boldsymbol{x}^*)$, when f is differentiable at x^*, is the directional derivative of f at $\boldsymbol{x}^*$ in the "direction" $(\boldsymbol{x} - \boldsymbol{x}^*)$.
Remark: A point $\boldsymbol{x}^*$ satisfying $(\star)$ is sometimes called a ***stationary point*** of f in C.

Problem 10. Let B be the unit closed ball centered at the origin given by

$$B = \left\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, \|\boldsymbol{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant 1 \right\},$$

and $f : B \to \mathbb{R}$ be a differentiable real-valued function. Consider the minimization problem $\min\limits_{\boldsymbol{x} \in B} f(\boldsymbol{x})$.

(1) Show that if f attains its minimum at $\boldsymbol{x}^* \in B$, then there exists $\lambda \leqslant 0$ such that

$$(\nabla f)(\boldsymbol{x}^*) = \lambda \boldsymbol{x}^*\,.$$

(2) Find the minimum of the function $f(x, y) = x^2 + 2y^2 - x$ on the unit closed disk centered at the origin $\left\{ (x, y) \,\middle|\, x^2 + y^2 \leqslant 1 \right\}$ using (1).

Problem 11. Let $\boldsymbol{a} \in \mathbb{R}^3$ be a vector, $b \in \mathbb{R}$, and C be a half plane given by

$$C = \left\{ \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \,\middle|\, \boldsymbol{a} \cdot \boldsymbol{x} \leqslant b \right\},$$

and $f : C \to \mathbb{R}$ be a differentiable real-valued function. Consider the minimization problem $\min\limits_{\boldsymbol{x} \in C} f(\boldsymbol{x})$. Show that if f attains its minimum at $\boldsymbol{x}^* \in C$, then there exists $\lambda \leqslant 0$ such that

$$(\nabla f)(\boldsymbol{x}^*) = \lambda \boldsymbol{a}\,.$$