Exercise Problem Sets 11

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Problem 1. Let f be a differentiable function and consider the surface $z = xf\big(\frac{y}{x}\big)$. Show that the tangent plane at any point (x_0, y_0, z_0) on the surface passes through the origin.

Problem 2. Prove that the angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) satisfies

$$\cos\theta = \frac{1}{\sqrt{f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2 + 1}}\,.$$

Problem 3. In the following problems, find all relative extrema and saddle points of the function. Use the Second Partials Test when applicable.

(1) $f(x, y) = x^2 - xy - y^2 - 3x - y$ (2) $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1$

(3) $f(x, y) = xy - 2x - 2y - x^2 - y^2$ (4) $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

(5) $f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$ (6) $f(x, y) = \frac{1}{x} + xy + \frac{1}{y}$

(7) $f(x, y) = \ln(x + y) + x^2 - y$ (8) $f(x, y) = 2\ln x + \ln y - 4x - y$

(9) $f(x, y) = xy\exp\big(-\frac{x^2 + y^2}{2}\big)$ (10) $f(x, y) = xy + e^{-xy}$

(11) $f(x, y) = (x^2 + y^2)e^{-x}$ (12) $f(x, y) = \Big(\frac{1}{2} - x^2 + y^2\Big)\exp(1 - x^2 - y^2)$

Problem 4. In the following problems, find the absolute extrema of the function over the region R (which contains boundaries).

(1) $f(x, y) = x^2 + xy$, and $R = \big\{(x, y)\,\big|\,|x| \leqslant 2, |y| \leqslant 1\big\}$

(2) $f(x, y) = 2x - 2xy + y^2$, and R is the region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 1$.

(3) $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}$, and $R = \big\{(x, y)\,\big|\,0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\big\}$.

(4) $f(x, y) = xy^2$, and $R = \big\{(x, y)\,\big|\,x \geqslant 0, y \geqslant 0, x^2 + y^2 \leqslant 3\big\}$.

(5) $f(x, y) = 2x^3 + y^4$, and $R = \big\{(x, y)\,\big|\,x^2 + y^2 \leqslant 1\big\}$.

Problem 5. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that the discriminant $f_{xx}f_{yy} - f_{xy}^2 = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point

Problem 6. Show that $f(x,y) = x^2 y e^{-x^2-y^2}$ has maximum values at $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and minimum values at $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$. Show also that f has infinitely many other critical points and the discriminant $f_{xx}f_{yy} - f_{xy}^2 = 0$ at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

Problem 7. Find two numbers a and b with $a \leqslant b$ such that

$$\int_a^b \sqrt[3]{24 - 2x - x^2}\, dx$$

has its largest value.

Problem 8. Let $m > n$ be natural numbers, and A be an $m \times n$ real matrix, $\boldsymbol{b} \in \mathbb{R}^m$ be a vector.

(1) Show that if the minimum of the function $f(x_1, \cdots, x_n) = \|A\boldsymbol{x} - \boldsymbol{b}\|$ occurs at the point $\boldsymbol{c} = (c_1, \cdots, c_n)$, then $\boldsymbol{c}$ satisfies $A^{\mathrm{T}}A\boldsymbol{c} = A^{\mathrm{T}}\boldsymbol{b}$.

(2) Find the relation between the linear regression and (1).

Problem 9. Let $\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$ be n points with $x_i \neq x_j$ if $i \neq j$. Use the Second Partials Test to verify that the formulas for a and b given by

$$a = \frac{n\sum\limits_{i=1}^n x_i y_i - \Big(\sum\limits_{i=1}^n x_i\Big)\Big(\sum\limits_{i=1}^n y_i\Big)}{n\sum\limits_{i=1}^n x_i^2 - \Big(\sum\limits_{i=1}^n x_i\Big)^2} \quad \text{and} \quad b = \frac{1}{n}\Big(\sum_{i=1}^n y_i - a\sum_{i=1}^n x_i\Big)$$

indeed minimize the function $S(a,b) = \sum\limits_{i=1}^n (ax_i + b - y_i)^2$.

Problem 10. The Shannon index (sometimes called the Shannon-Wiener index or Shannon-Weaver index) is a measure of diversity in an ecosystem. For the case of three species, it is defined as

$$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3\,,$$

where p_i is the proportion of species i in the ecosystem.

(1) Express H as a function of two variables using the fact that $p_1 + p_2 + p_3 = 1$.

(2) What is the domain of H?

(3) Find the maximum value of H. For what values of p_1, p_2, p_3 does it occur?

Problem 11. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq\,,$$

where p, q, and r are the proportions of A, B, and O in the population. Use the fact that $p+q+r = 1$ to show that P is at most $\frac{2}{3}$.

Problem 12. Find an equation of the plane that passes through the point $(1, 2, 3)$ and cuts off the smallest volume in the first octant.