Exercise Problem Sets 9

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Problem 1. Define

$$f(x,y) = \begin{cases} x^2 \arctan \dfrac{y}{x} - y^2 \arctan \dfrac{x}{y} & \text{if } x, y \neq 0\,, \\ 0 & \text{if } x = 0 \text{ or } y = 0\,. \end{cases}$$

Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

Problem 2. Show that each of the following functions is not differentiable at the origin.

(1) $f(x,y) = \sqrt{x}\cos y$ (2) $f(x,y) = \sqrt{|xy|}$

Problem 3. In the following, show that both $f_x(0,0)$ and $f_y(0,0)$ both exist but that f is not differentiable at $(0,0)$.

(1) $f(x,y) = \begin{cases} \dfrac{5x^2y}{x^3+y^3} & \text{if } x^3+y^3 \neq 0\,, \\ 0 & \text{if } x^3+y^3 = 0\,. \end{cases}$ (2) $f(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$

(3) $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$ (4) $f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$

Problem 4. Let $f, g : (a,b) \to \mathbb{R}$ be real-valued function, $h(x,y) = f(x)g(y)$, and $c, d \in (a,b)$. Show that if f is differentiable at c and g is differentiable at d, then h is differentiable at (c,d).

Problem 5. Show that the function $f(x,y) = \sqrt{x^2+y^2}\sin\sqrt{x^2+y^2}$ is differentiable at $(0,0)$.

Problem 6. Investigate the differentiability of the following functions at the point $(0,0)$.

(1) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ (2) $f(x,y) = \begin{cases} \dfrac{xy}{x+y^2} & \text{if } x+y^2 \neq 0\,, \\ 0 & \text{if } x+y^2 = 0 \end{cases}$

(3) $f(x,y) = \begin{cases} (x^2+y^2)\sin \dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$