Exercise Problem Sets 7（部份解答）

Apr. 30. 2020

Problem 6. Let C_1 be the polar graph of the polar function $r = 1 + \cos\theta$ (which is a cardioid), and C_2 be the polar graph of the polar function $r = 3\cos\theta$ (which is a circle). See the following figure for reference.

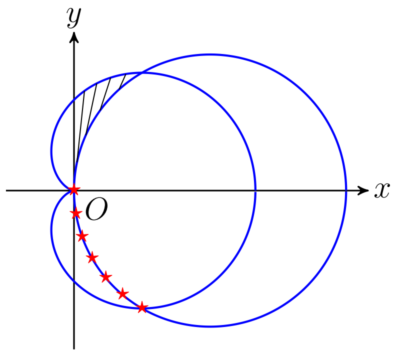


Figure 1: The polar graphs of the polar equations $r = 1 + \cos\theta$ and $r = 3\cos\theta$

(1) Find the intersection points of C_1 and C_2.

(2) Find the line L passing through the lowest intersection point and tangent to the curve C_2.

(3) Identify the curve marked by $\star$ on the θr-plane for $0 \leqslant \theta \leqslant 2\pi$.

(4) Find the area of the shaded region.

Solution. (1) Let $1 + \cos\theta = 3\cos\theta$. Then $2\cos\theta = 1$ or $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$. From the figure, it is also clear that C_1 and C_2 intersection at the origin. Therefore, the points of intersections are

$$\Big(\frac{3}{4}, \frac{3\sqrt{3}}{4}\Big), \quad \Big(\frac{3}{4}, -\frac{3\sqrt{3}}{4}\Big), \quad (0,0).$$

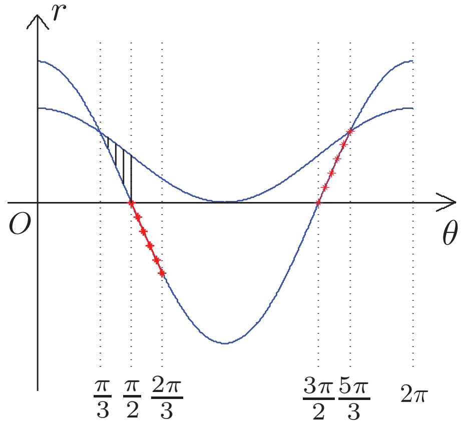
(2) C_2 can be parametrized by $\big\{(x,y) \in \mathbb{R}^2 \,\big|\, x = 3\cos^2\theta, y = 3\cos\theta\sin\theta\big\}$. Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3\sin^2\theta + 3\cos^2\theta}{-6\cos\theta\sin\theta} = \frac{\sin^2\theta - \cos^2\theta}{2\cos\theta\sin\theta};$$

thus at the lowest point of intersection $\big(\theta = \dfrac{5\pi}{3}\big)$, $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{3}}$. As a consequence, the desired tangent line is

$$y = -\frac{1}{\sqrt{3}}\Big(x - \frac{3}{4}\Big) - \frac{3\sqrt{3}}{4} = -\frac{x}{\sqrt{3}} - \frac{\sqrt{3}}{2}.$$

(3) The curve marked by $\star$ is in the fourth quadrant, on the circle $r = 3\cos\theta$ with end-points $(0,0)$ and $\Big(\dfrac{3}{4}, -\dfrac{3\sqrt{3}}{4}\Big)$. Therefore, it corresponds to the curves marked by $\star$ shown in the following figure.



(4) The shaded region on xy-plane corresponds to the shaded region in Figure 2. Therefore, the area of the shaded region (on xy-plane) is

$$\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \Big[(1+\cos\theta)^2 - 9\cos^2\theta\Big]\,d\theta = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \Big[1+2\cos\theta - 4(1+\cos 2\theta)\Big]\,d\theta$$

$$= \frac{-3\theta + 2\sin\theta - 2\sin 2\theta}{2}\Big|_{\theta=\frac{\pi}{3}}^{\theta=\frac{\pi}{2}} = 1 - \frac{\pi}{4}. \qquad \square$$

以下問題部份小題要用到第七章的觀念，為了題目的完整性一併呈現，但是與第十二章有關的是(1)(2)(3) 三小題

Problem 7. Let R be the region bounded by the circle $r = 1$ and outside the lemniscate $r^2 = -2\cos 2\theta$, and is located on the right half plane (see the shaded region in the graph).

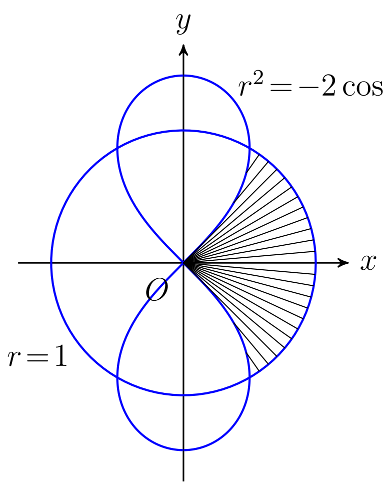


Figure 2: The polar graphs of the polar equations $r = 1$ and $r^2 = -2\cos 2\theta$

(1) Find the points of intersection of the circle $r = 1$ and the lemniscate $r^2 = -2\cos 2\theta$.

(2) Show that the straight line $x = \dfrac{1}{2}$ is tangent to the lemniscate at the points of intersection on the right half plane.

(3) Find the area of R.

(4) Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

(a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{0.1}$$

for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) Solving (0.1), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1-4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = \pi \int_0^{\frac{1}{2}} c(x)dx + \pi \int_{\frac{1}{2}}^1 d(x)dx.$$

Compute $\displaystyle\int_{\frac{1}{2}}^1 \Big[d(x) - (1-x^2)\Big]dx$.

(c) Evaluate I by first computing the integral $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1-4x^2}dx$, and then find I.

(5) Find the area of the surface of revolution obtained by rotating the boundary of R about the x-axis.

Solution. (1) Let $2\cos 2\theta = -1$, then $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$; thus the points of intersection are

$$\Big(\frac{1}{2}, \frac{\sqrt{3}}{2}\Big), \Big(\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big), \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big), \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\Big).$$

(2) On the lemniscate, $r = \pm\sqrt{-2\cos 2\theta}$; thus

$$\frac{dx}{d\theta}\Big|_{\theta=\frac{\pi}{3}} = \Big[r'(\theta)\cos\theta - r(\theta)\sin\theta\Big]\Big|_{\theta=\frac{\pi}{3}} = \sqrt{2}\Big[\frac{\sin 2\theta}{\sqrt{-\cos 2\theta}}\cos\theta - \sqrt{-\cos 2\theta}\sin\theta\Big]\Big|_{\theta=\frac{\pi}{3}} = 0.$$

Similar computation shows that $\frac{dx}{d\theta}\Big|_{\theta=\frac{2\pi}{3}} = 0$; thus $x = \frac{1}{2}$ is tangent to the lemniscate.

(3) The area of the shaded region is

$$2 \times \frac{1}{2}\Big[\int_0^{\frac{\pi}{4}} 1^2 d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 + 2\cos 2\theta)d\theta\Big] = \frac{\pi}{4} + \big(\theta + \sin 2\theta\big)\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1.$$

(4) If (x, y) is on the lemniscate, then

$$x^2 + y^2 = -2\Big(2\frac{x^2}{x^2+y^2} - 1\Big) = \frac{2(y^2 - x^2)}{x^2 + y^2}$$

which implies that

$$y^4 + 2(x^2 - 1)y^2 + x^4 + 2x^2 = 0.$$

Therefore,

$$y^2 = -(x^2-1) + \sqrt{(x^2-1)^2 - (x^4+2x^2)} = 1 - x^2 + \sqrt{1-4x^2}.$$

Therefore, the volume of the solid of revolution obtained by rotating R about the y-axis is

$$\begin{aligned}
&\pi \int_0^{\frac{1}{2}} \Big[1 - x^2 + \sqrt{1-4x^2}\Big] dx + \pi \int_{\frac{1}{2}}^1 (1-x^2)dx \\
&\qquad = \pi \int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx + \pi \int_0^1 (1-x^2)dx \\
&\qquad = \pi \int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx + \pi\Big(x - \frac{x^3}{3}\Big)\Big|_{x=0}^{x=1} = \pi \int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx + \frac{2\pi}{3}.
\end{aligned}$$

On the other hand, the integral can be evaluated by making a change of variable $x = \dfrac{\sin\theta}{2}$:

$$\begin{aligned}
\int \sqrt{1-4x^2}\, dx &= \frac{1}{2}\int \cos^2\theta d\theta = \frac{1}{4}\int \big(1+\cos 2\theta\big)d\theta \\
&= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{4}\big(\theta + \sin\theta\cos\theta\big) + C \\
&= \frac{1}{4}\Big(\sin^{-1} 2x + 2x\sqrt{1-4x^2}\,\Big) + C.
\end{aligned}$$

Therefore,

$$\int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx = \frac{\pi}{8}$$

and the volume of the solid of revolution obtained by rotating R about the x-axis is $\dfrac{2\pi}{3} + \dfrac{\pi^2}{8}$.

(5) There are two parts of the surface: one from rotating the lemniscate and the other from rotating the sphere. The area obtained by rotating the part of the lemniscate is

$$\begin{aligned}
\int 2\pi|y|ds &= \int 2\pi|r\sin\theta|\sqrt{r'^2+r^2}d\theta \\
&= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{-2\cos 2\theta}\sin\theta \sqrt{(\sqrt{-2\cos 2\theta}')^2 + (-\sqrt{2\cos 2\theta})^2 d\theta} \\
&= 4\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{-\cos 2\theta}\sin\theta \frac{1}{\sqrt{-\cos 2\theta}}\, d\theta \\
&= 4\pi(-\cos\theta)\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} = 2\pi(\sqrt{2}-1).
\end{aligned}$$

The part obtained by rotating the part of the sphere is

$$\int 2\pi|y|ds = 2\pi \int_0^{\frac{\pi}{3}} \sin\theta\sqrt{1'^2+1^2}d\theta = 2\pi(-\cos\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{3}} = \pi.$$

The total area is then $(2\sqrt{2}-1)\pi$. □

Problem 8. Let R be the region bounded by the lemniscate $r^2 = 2\cos 2\theta$ and is outside the circle $r = 1$ (see the shaded region in the graph).

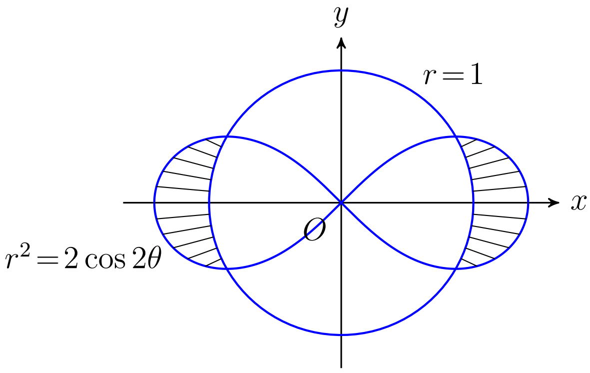


Figure 3: The polar graphs of the polar equations $r^2 = 2\cos 2\theta$ and $r = 1$

(1) Find the area of R.

(2) Find the slope of the tangent line passing thought the point on the lemniscate corresponding to $\theta = \frac{\pi}{6}$.

(3) Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

(a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{0.2}$$

for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) Solving (0.2), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1+4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = 2 \times \Big[\pi \int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} c(x)dx - \pi \int_{\frac{\sqrt{3}}{2}}^{1} d(x)dx\Big].$$

Compute $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{1} \big[d(x) - (1 - x^2)\big]dx$.

(c) Evaluate I by first computing the integral $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} \sqrt{1+4x^2}dx$, and then find I.

(4) Find the surface area of the surface of revolution obtained by rotating the boundary of R about the x-axis.

Solution. (1) First we find the points of intersection: let $2\cos 2\theta = 1$, then $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$; thus the points of intersection are

$$\Big(\frac{\sqrt{3}}{2}, \frac{1}{2}\Big), \Big(-\frac{\sqrt{3}}{2}, \frac{1}{2}\Big), \Big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big), \Big(\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big).$$

The area of the desired region is

$$4\int_0^{\frac{\pi}{6}} (2\cos 2\theta - 1)d\theta = 4\big(\sin 2\theta - \theta\big)\Big|_{\theta=0}^{\theta=\frac{\pi}{6}} = 4\Big(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\Big) = 2\sqrt{3} - \frac{2\pi}{3}.$$

(3) If (x,y) is on the lemniscate, then

$$x^2+y^2=2\Big(2\frac{x^2}{x^2+y^2}-1\Big)=\frac{2(x^2-y^2)}{x^2+y^2}$$

which implies that

$$y^4+2(x^2+1)y^2+x^4-2x^2=0.$$

Therefore,

$$y^2=-(x^2+1)+\sqrt{(x^2+1)^2-(x^4-2x^2)}=-(x^2+1)+\sqrt{1+4x^2}.$$

Therefore, the volume of the solid of revolution obtained by rotating R about the x-axis is

$$\begin{aligned}
2\pi\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}&\Big[-(x^2+1)+\sqrt{1+4x^2}\Big]dx-2\pi\int_{\frac{\sqrt{3}}{2}}^{1}(1-x^2)dx\\
&=2\pi\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\sqrt{1+4x^2}\,dx-2\pi\Big(\frac{x^3}{3}+x\Big)\Big|_{x=\frac{\sqrt{3}}{2}}^{x=\sqrt{2}}-2\pi\Big(x-\frac{x^3}{3}\Big)\Big|_{x=\frac{\sqrt{3}}{2}}^{x=1}\\
&=2\pi\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\sqrt{1+4x^2}\,dx+2\pi\Big(\frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{8}-\frac{2\sqrt{2}}{3}-\sqrt{2}-1+\frac{1}{3}\Big)\\
&=2\pi\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\sqrt{1+4x^2}\,dx+2\pi\Big(\sqrt{3}-\frac{5\sqrt{2}}{3}-\frac{2}{3}\Big).
\end{aligned}$$

On the other hand, the integral can be evaluated by making a change of variable $x=\dfrac{\tan\theta}{2}$:

$$\begin{aligned}
\int\sqrt{1+4x^2}\,dx&=\frac{1}{2}\int\sec^3\theta d\theta=\frac{1}{2}\int\sec\theta\big(\tan^2\theta+1\big)d\theta\\
&=\frac{1}{2}\int\tan\theta d\sec\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|\\
&=\frac{1}{2}\Big[\tan\theta\sec\theta-\int\sec^3\theta d\theta\Big]+\frac{1}{2}\ln|\sec\theta+\tan\theta|\\
&=-\frac{1}{2}\int\sec^3\theta d\theta+\frac{1}{2}\Big[\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|\Big]\\
&=\frac{1}{4}\Big[\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|\Big]+C\\
&=\frac{1}{4}\Big[2x\sqrt{1+4x^2}+\ln|2x+\sqrt{1+4x^2}|\Big]+C.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\sqrt{1+4x^2}\,dx&=\frac{1}{4}\Big[6\sqrt{2}+\ln(3+2\sqrt{2})-2\sqrt{3}-\ln(2+\sqrt{3})\Big]\\
&=\pi\Big(\frac{3\sqrt{2}}{2}-\frac{\sqrt{3}}{2}+\frac{1}{4}\ln\frac{3+2\sqrt{2}}{2+\sqrt{3}}\Big)
\end{aligned}$$

and the volume of the solid of revolution obtained by rotating R about the x-axis is

$$\pi\Big(\frac{1}{2}\ln\frac{3+2\sqrt{2}}{2+\sqrt{3}}+\sqrt{3}-\frac{\sqrt{2}}{3}-\frac{4}{3}\Big).$$

(4) There are two parts of the surface: one from rotating the lemniscate and the other from rotating the sphere. The area obtained by rotating the part of the lemniscate is

$$\begin{aligned}\int 2\pi y ds &= \int 2\pi r \sin\theta\sqrt{r'^2 + r^2}d\theta \\ &= 2\pi \int_0^{\frac{\pi}{6}} \sqrt{2\cos 2\theta}\sin\theta\sqrt{(\sqrt{2\cos 2\theta}')^2 + (\sqrt{2\cos 2\theta})^2}d\theta \\ &= 4\pi \int_0^{\frac{\pi}{6}} \sqrt{\cos 2\theta}\sin\theta\frac{1}{\sqrt{\cos 2\theta}}\, d\theta \\ &= 4\pi(-\cos\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{6}} = 4\pi\Big(1 - \frac{\sqrt{3}}{2}\Big).\end{aligned}$$

The part obtained by rotating the part of the sphere is

$$\int 2\pi y ds = 2\pi \int_0^{\frac{\pi}{6}} \sin\theta\sqrt{1'^2 + 1^2}d\theta = 2\pi\Big(1 - \frac{\sqrt{3}}{2}\Big).$$

The total area is then $12\pi\Big(1 - \frac{\sqrt{3}}{2}\Big)$. □

Problem 9. Let C_1, C_2 be the curves given by polar coordinate $r = 1 - 2\sin\theta$ and $r = 4 + 4\sin\theta$, respectively, and the graphs of C_1 and C_2 are given in Figure 4.

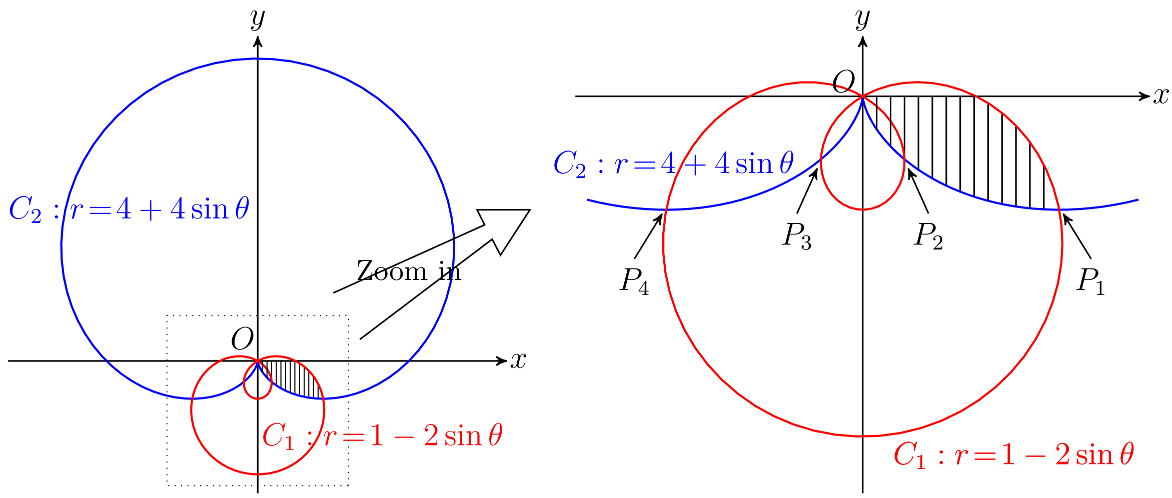


Figure 4: The polar graphs of the polar equations $r = 1 - 2\sin\theta$ and $r = 4 + 4\sin\theta$

(1) Let $P_1, \cdots, P_4$ be four points of intersection of curves C_1 and C_2 as shown in Figure 4 (the fifth one is the origin). What are the Cartesian coordinates of P_1 and P_2?

(2) Let L_1 and L_2 be two straight lines passing P_1 and tangent to C_1, C_2, respectively. Find the cosine value of the acute/smaller angle between L_1 and L_2.

(3) Compute the area of the shaded region.

Solution. Some of the important correspondence between curves and points in (x, y)-plane and (r, θ)-plane are shown in the following figures.

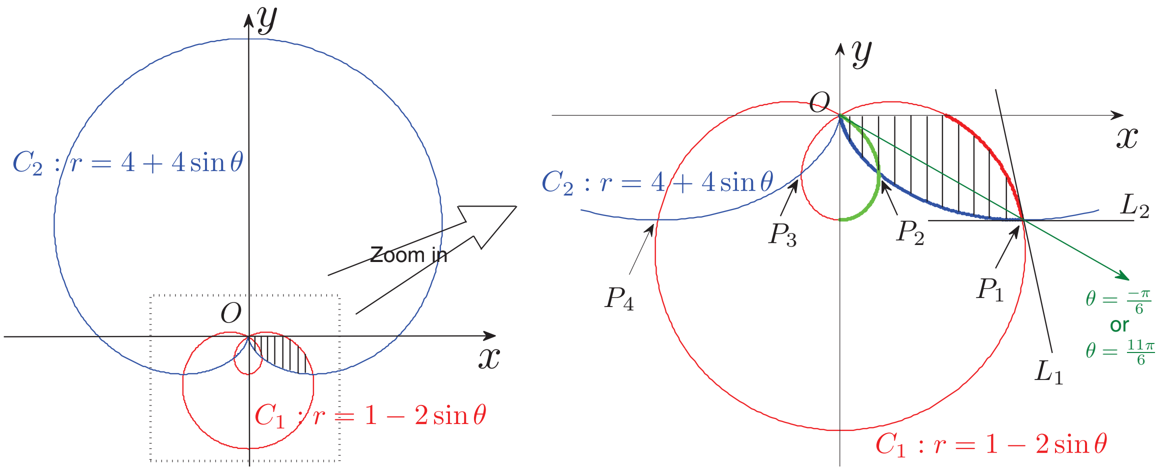


Figure 5: The polar graphs of the polar equations $r = 1 - 2\sin\theta$ and $r = 4 + 4\sin\theta$

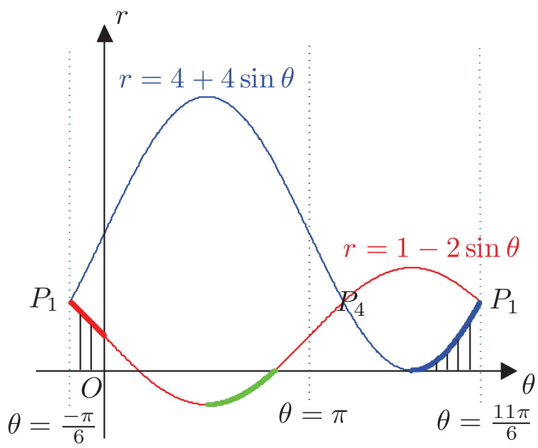


Figure 6: The graph of $r = 1 - 2\sin\theta$ and $r = 4 + 4\sin\theta$ on the θr-plane

(1) The point P_1 is obtained by solving $1 - 2\sin\theta = 4 + 4\sin\theta$ which gives us $\sin\theta = -0.5$. Therefore, $\theta = -\dfrac{\pi}{6}$ or $\theta = \dfrac{11\pi}{6}$. The Cartesian coordinate of P_1 is then

$$\left((1 - 2\sin\frac{-\pi}{6})\cos\frac{-\pi}{6}, (1 - 2\sin\frac{-\pi}{6})\sin\frac{-\pi}{6}\right) = (\sqrt{3}, -1)\,.$$

It is a bit tricky to find P_2. As one can see in Figure 6, the thick green curve does not intersect the thick blue curve in the (r, θ)-plane, while these two curves still intersect in (x, y)-plane. The reason for this phenomena is due to the non-unique representation of curves in polar coordinates. In fact, $r = f(\theta)$ and $r = -f(\theta + \pi)$ will present the same curve. We rewrite $r(\theta) = 1 - 2\sin\theta$ as $r(\theta) = -(1 - 2\sin(\theta + \pi)) = -1 - 2\sin\theta$. Then the point P_2 is obtained by solving $-1 - 2\sin\theta = 4 + 4\sin\theta$ which gives $\sin\theta = -\dfrac{5}{6}$ (hence $\cos\theta = \pm\dfrac{\sqrt{11}}{6}$). Therefore, the Cartesian coordinate of P_2 is

$$\left((1 - 2\cdot\frac{-5}{6})\frac{\sqrt{11}}{6}, (1 - 2\cdot\frac{-5}{6})\frac{-5}{6}\right) = \left(\frac{4\sqrt{11}}{9}, -\frac{20}{9}\right).$$

(2) Let the green line denote $\theta = \dfrac{-\pi}{6}$ or $\theta = \dfrac{11\pi}{6}$. Then the green line divides the shaded region into two pieces, and each piece corresponds to one of two shaded regions in the (r, θ)-plane.

Therefore, the area of the shaded region is

$$\begin{aligned}
&\frac{1}{2}\int_{-\frac{\pi}{6}}^{0}(1-2\sin\theta)^2 d\theta+\frac{1}{2}\int_{\frac{3\pi}{2}}^{\frac{11\pi}{6}}(4+4\sin\theta)^2 d\theta\\
&\qquad=\frac{1}{2}\int_{-\frac{\pi}{6}}^{0}(1-4\sin\theta+4\sin^2\theta)d\theta+8\int_{\frac{3\pi}{2}}^{\frac{11\pi}{6}}(1+2\sin\theta+\sin^2\theta)d\theta\,.
\end{aligned}$$

Using the formula $\sin^2\theta=\dfrac{1-\cos 2\theta}{2}$ and $\cos^2\theta=\dfrac{1+\cos 2\theta}{2}$, we know that the area of interest is

$$\begin{aligned}
&\frac{1}{2}\int_{-\frac{\pi}{6}}^{0}(3-4\sin\theta-2\cos 2\theta)d\theta+4\int_{\frac{3\pi}{2}}^{\frac{11\pi}{6}}(3+4\sin\theta-\cos 2\theta)d\theta\\
&\qquad=\frac{3\theta+4\cos\theta-\sin 2\theta}{2}\Big|_{\theta=-\frac{\pi}{6}}^{\theta=0}+4(3\theta-4\cos\theta-\frac{\sin 2\theta}{2})\Big|_{\theta=\frac{3\pi}{2}}^{\theta=\frac{11\pi}{6}}\\
&\qquad=\Big(\frac{\pi}{4}+2-\frac{5\sqrt{3}}{4}\Big)+(4\pi-7\sqrt{3})=\frac{17\pi}{4}+2-\frac{33\sqrt{3}}{4}\,.
\end{aligned}$$

(3) Let

$$\begin{aligned}
(x_1(\theta),y_1(\theta))&=\big((1-2\sin\theta)\cos\theta,(1-2\sin\theta)\sin\theta\big)\,,\\
(x_2(\theta),y_2(\theta))&=\big((4+4\sin\theta)\cos\theta,(4+4\sin\theta)\sin\theta\big)\,.
\end{aligned}$$

Then

$$\begin{aligned}
(x_1'(\theta),y_1'(\theta))&=(-2\cos^2\theta-(1-2\sin\theta)\sin\theta,-2\sin\theta\cos\theta+(1-2\sin\theta)\cos\theta)\\
&=(-2\cos 2\theta-\sin\theta,-2\sin 2\theta+\cos\theta)\,,\\
(x_2'(\theta),y_2'(\theta))&=(4\cos^2\theta-(4+4\sin\theta)\sin\theta,4\sin\theta\cos\theta+(4+4\sin\theta)\cos\theta)\\
&=(4\cos 2\theta-4\sin\theta,4\sin 2\theta+4\cos\theta)\,;
\end{aligned}$$

hence the "direction" of C_1 and C_2 are given by

$$\Big(x_1'(-\frac{\pi}{6}),y_1'(-\frac{\pi}{6})\Big)=\Big(-\frac{1}{2},\frac{3\sqrt{3}}{2}\Big)\,,\qquad\Big(x_2'(-\frac{\pi}{6}),y_2'(-\frac{\pi}{6})\Big)=(4,0)\,.$$

Therefore, if the smaller angle between L_1 and L_2 is φ, then

$$\cos\varphi=\frac{\Big|(4,0)\cdot(-\frac{1}{2},\frac{3\sqrt{3}}{2})\Big|}{\|(4,0)\|\|(-\frac{1}{2},\frac{3\sqrt{3}}{2})\|}=\frac{2}{4\sqrt{7}}=\frac{\sqrt{7}}{14}\,.$$

□