Exercise Problem Sets 6

Apr. 17. 2020

Problem 1. In class we have introduced the permutation symbol ε_{ijk} and use it to define the cross product: for two given vectors $\mathbf{u} = u_1\mathbf{i} + u_2\,\mathbf{j} + u_3\mathbf{k} = \sum\limits_{i=1}^{3} u_i\mathbf{e}_i$ and $\mathbf{v} = v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k} = \sum\limits_{i=1}^{3} v_i\mathbf{e}_i$, the cross product $\mathbf{u} \times \mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = \sum_{i=1}^{3} \Big(\sum_{j,k=1}^{3} \varepsilon_{ijk} u_j v_k \Big) \mathbf{e}_i = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_j v_k \mathbf{e}_i\,.$$

Use the summation notation above without expanding the sum (不要展開成向量和的形式，直接用 Σ 操作) and the identity

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$$

to prove the following.

(1) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in space. (Is the associative law $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ true?)

(2) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$ for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in space.

Problem 2.

(1) Let $\mathbf{u}, \mathbf{v}$ be vectors in space satisfying $\mathbf{u} \cdot \mathbf{v} = \sqrt{3}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{i} + 2\,\mathbf{j} + 2\mathbf{k}$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

(2) Let $\mathbf{u}, \mathbf{v}$ be vectors in space. What can you conclude if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = 0$?

(3) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in space. Show that if $\mathbf{u} \neq \mathbf{0}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

Problem 3.

(1) Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is

$$d = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|}\,,$$

where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.

(2) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{\|\mathbf{a} \times \mathbf{b}\|}\,,$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$ and $\mathbf{c} = \overrightarrow{QP}$.

Problem 4. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

Problem 5. Replace the polar equations in the following questions with equivalent Cartesian equations.

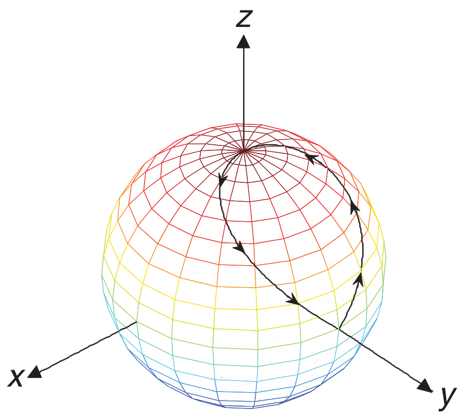
(1) $r^2 \sin 2\theta = 2$ (2) $r = 4\tan\theta\sec\theta$ (3) $r = \csc\theta e^{r\cos\theta}$ (4) $r\sin\theta = \ln r + \ln\cos\theta$.

Problem 6. Let C be a smooth curve parameterized by

$$\boldsymbol{r}(t) = (\cos t\sin t, \sin t\sin t, \cos t), \qquad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

(1) Show that C is a closed curve on the unit sphere $\mathbb{S}^2$.

(2) Using the spherical coordinate, the curve C above corresponds to a curve on the $\theta\phi$-plane. Find the curve in the region $\left\{(\theta,\phi)\,\middle|\,0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \phi \leqslant \pi\right\}$.



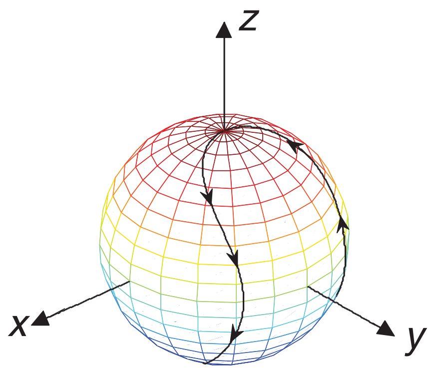
Remark: 想像球面是地球，有人開飛機飛行了 C 這個路線。這個路線在世界地圖上對應到另一個曲線，第二小題即是要求在世界地圖上這個曲線為何。

Problem 7. Let C be a smooth curve parameterized by

$$\boldsymbol{r}(t) = \big(\cos(\sin t)\sin t, \sin(\sin t)\sin t, \cos t\big), \qquad t \in [0, 2\pi].$$

(1) Show that C is a closed curve on the unit sphere $\mathbb{S}^2$.

(2) Using the spherical coordinate, the curve C above corresponds to a curve on the $\theta\phi$-plane. Find the curve in the region $\left\{(\theta,\phi)\,\middle|\,0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \phi \leqslant \pi\right\}$.



Problem 8. In class we talked about how to find the total distance that you travel when you walk along a path according to the position vector $\boldsymbol{r} : [a, b] \to \mathbb{R}^2$. The total distance travelled can be computed by

$$\int_a^b \|\boldsymbol{r}'(t)\| \, dt$$

when $\boldsymbol{r}$ is continuously differentiable. Complete the following.

1. Let $\boldsymbol{r} : [0, 4\pi] \to \mathbb{R}^2$ be given by $\boldsymbol{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$. Find the image of $[0, 4\pi]$ under $\boldsymbol{r}$.
2. Compute the integral $\displaystyle\int_0^{4\pi} \|\boldsymbol{r}'(t)\| \, dt$. Does it agree with the length of the curve $C \equiv \boldsymbol{r}\big([0, 4\pi]\big)$?

Problem 9. To illustrate that the length of a smooth space curve does not depend on the parametrization you use to compute it, calculate the length of one turn of the helix in Example 1 with the following parametrizations.

1. $\boldsymbol{r}(t) = \cos(4t)\mathbf{i} + \sin(4t)\,\mathbf{j} + 4t\mathbf{k}$, $t \in \left[0, \frac{\pi}{2}\right]$.
2. $\boldsymbol{r}(t) = \cos\frac{t}{2}\mathbf{i} + \sin\frac{t}{2}\mathbf{j} + \frac{t}{2}\mathbf{k}$, $t \in [0, 4\pi]$.
3. $\boldsymbol{r}(t) = \cos t\mathbf{i} - \sin t\,\mathbf{j} - t\mathbf{k}$, $t \in [-2\pi, 0]$.