• Area of regions enclosed by simple closed curves

Let C be a simple closed curve in the plane parameterized by $\boldsymbol{r} : [a,b] \to \mathbb{R}^2$. Suppose that

1. $\boldsymbol{r}(t) = \big(x(t), y(t)\big)$ moves **counter-clockwise** (that is, the region enclosed by C is on the left-hand side when moving along C) as t increases.

2. There exists $c \in (a,b)$ such that x is strictly decreasing on $[a,c]$ and is strictly increasing on $[c,b]$.

3. $x'y$ is Riemann integrable on $[a,b]$ (for example, x is continuously differentiable on $[a,b]$).

Based on the assumption above, in the following we "prove" that

$$\text{the area of the region enclosed by } C \text{ is } -\int_a^b x'(t)y(t)\,dt\,. \tag{0.1}$$

Let $\varepsilon > 0$ be given. Since $x'y$ is Riemann integrable on $[a,b]$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$, then

$$\text{any Riemann sum of } x'y \text{ for } \mathcal{P} \text{ lies in } \Big(\int_a^b x'(t)y(t)\,dt - \varepsilon, \int_a^b x'(t)y(t)\,dt + \varepsilon\Big).$$

Let $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = c = s_0 < s_1 < \cdots < s_n = b\}$ be a partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$ and $x(t_{n-k}) = x(s_k) \equiv x_k$ for all $0 \leqslant k \leqslant n$. By the mean value theorem, for each $1 \leqslant k \leqslant n$ there exist $c_k \in [t_{k-1}, t_k]$ and $d_k \in [s_{k-1}, s_k]$ such that

$$x(t_k) - x(t_{k-1}) = x'(c_k)(t_k - t_{k-1}) \quad \text{and} \quad x(s_k) - x(s_{k-1}) = x'(d_k)(s_k - s_{k-1})\,. \tag{0.2}$$

Then

$$\Big|\sum_{k=1}^n x'(c_k)y(c_k)(t_k - t_{k-1}) + \sum_{k=1}^n x'(d_k)y(d_k)(s_k - s_{k-1}) - \int_a^b x'(t)y(t)\,dt\Big| < \varepsilon\,.$$

Using (0.2), we find that

$$\begin{aligned}
&\Big|\sum_{k=1}^n (x_k - x_{k-1})\big[y(c_{n-k+1}) - y(d_k)\big] - \Big(-\int_a^b x'(t)y(t)\Big)\,dt\Big| \\
&\quad = \Big|\sum_{k=1}^n (x_{k-1} - x_k)y(c_{n-k+1}) + \sum_{k=1}^n (x_k - x_{k-1})y(d_k) - \int_a^b x'(t)y(t)\,dt\Big| \\
&\quad = \Big|\sum_{k=1}^n (x_{n-k} - x_{n-k+1})y(c_k) + \sum_{k=1}^n (x_k - x_{k-1})y(d_k) - \int_a^b x'(t)y(t)\,dt\Big| \\
&\quad = \Big|\sum_{k=1}^n \big(x(t_k) - x(t_{k-1})\big)y(c_k) + \sum_{k=1}^n \big(x(s_k) - x(s_{k-1})\big)y(d_k) - \int_a^b x'(t)y(t)\,dt\Big| < \varepsilon\,.
\end{aligned}$$

Since $\sum\limits_{k=1}^n (x_k - x_{k-1})\big[y(c_{n-k+1}) - y(d_k)\big]$ is an approximation of the area of the region enclosed by C, (0.1) is concluded.

Similar argument can be applied to conclude that

$$\text{the area of the region enclosed by } C \text{ is } \int_a^b x(t)y'(t)\,dt\,. \tag{0.3}$$

if xy' is Riemann integrable on $[a,b]$. Combining (0.1) and (0.3), we obtain that

$$\text{the area of the region enclosed by } C \text{ is } \frac{1}{2}\int_a^b \Big[x(t)y'(t) - x'(t)y(t)\Big]\,dt\,. \tag{0.4}$$

Example 0.1. Let C be the curve parameterized by $\boldsymbol{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then clearly $\boldsymbol{r}$ satisfies condition 1-3. Therefore, the area of the region enclosed by C can be computed by the following three ways:

1. Using (0.1),

$$-\int_0^{2\pi} \frac{d\cos t}{dt}\sin t\,dt = \int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt = \frac{1}{2}\Big(t - \frac{\sin(2t)}{2}\Big)\Big|_{t=0}^{t=2\pi} = \pi\,.$$

2. Using (0.3),

$$\int_0^{2\pi} \cos t\,\frac{d\sin t}{dt}\,dt = \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt = \frac{1}{2}\Big(t + \frac{\sin(2t)}{2}\Big)\Big|_{t=0}^{t=2\pi} = \pi\,.$$

3. Using (0.4),

$$\frac{1}{2}\int_0^{2\pi} \Big(\cos t\frac{d\sin t}{dt} - \frac{d\cos t}{dt}\sin t\Big)\,dt = \frac{1}{2}\int_0^{2\pi} \big(\cos^2 t + \sin^2 t\big)\,dt = \frac{1}{2}\int_0^{2\pi} 1dt = \pi\,.$$