Calculus MA1001-A Midterm 2 Sample

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這份樣本題只需要用變數變換的技巧就能做出所有的問題，所以請同學盡量試著不用複雜的積分技巧做這份題目，這樣也許會對大家在遇到實際的考試題時有比較快速的解法。

Problem 1. 定義敘述、定理敘述與證明題。

Problem 2. Find $\dfrac{d}{dx}\displaystyle\int_{\ln x}^{\arctan x} 3^{-u^2}du$ for $x > 0$. (Fundamental Theorem of Calculus)

Problem 3. Find the limit $\displaystyle\lim_{x\to\infty} x\Big[\big(1+\frac{1}{x}\big)^x - e\Big]$. (L'Hôspital's rule)

Problem 4. Find the indefinite integral $\displaystyle\int \frac{8}{(e^x+e^{-x})^4}\,dx$. (Indefinite integrals)

Problem 5. Find the definite integral $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^3 x\,dx$ using the substitution of variable $t = \tan\dfrac{x}{2}$. (Integration by a given substitution)

Problem 6. Show that $e^x > 1+(1+x)\ln(1+x)$ for all $x > 0$. (Inequality)

Problem 7. Let $f(x) = x\ln x$. Compute $\displaystyle\int f(x)\,dx$ by completing the following.

1. Let $b > 1$, and $\mathcal{P} = \{1 = x_0 < x_1 < \cdots < x_n = b\}$, where $x_k = r^k$, be the "geometric" partition of $[1, b]$. Show that the Riemann sum of f for $\mathcal{P}$ using the right end-point rule is

$$I_n = (r-1)\ln r\sum_{i=1}^{n} kr^{2k-1}\,. \quad \text{(Riemann sums)}$$

2. Show that $\displaystyle\sum_{k=1}^{n} kr^{2k-1} = \frac{1}{(r^2-1)^2}\Big[nr^{2n+3}-(n+1)r^{2n+1}+r\Big]$. (Summation identities)

   Hint: Note that
$$\sum_{k=1}^{n} kr^{2k-1} = \frac{1}{2}\frac{d}{dr}\sum_{k=1}^{n} r^{2k}\,.$$
   Differentiate the identity above to conclude the identity.

3. Show that $\displaystyle\lim_{n\to\infty} I_n = \frac{1}{2}b^2\ln b + \frac{1-b^2}{4}$. (Integral as the limit of Riemann sums)

4. Find $\displaystyle\int x\ln x\,dx$. (Indefinite integrals from the definite integrals)

Problem 8. Let y be a function satisfy

$$x(1-x^2)^2\,y' - (1-x^2)^{\frac{3}{2}}y = x^2 \qquad \textit{and} \qquad y\big(\tfrac{1}{\sqrt{2}}\big) = 0\,.$$

Find $y\big(\frac{1}{2}\big)$. (Use integrating factor to solve first order differential equations)