Calculus MA1001-B Quiz 11

National Central University, Dec. 10 2019

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Problem 1. (2pts) State a version of L'Hôspital rule (write the version that you are going to apply in Problem 2 if possible).

Solution. Let $f, g : (a, \infty) \to \mathbb{R}$ be differentiable functions, and $\frac{f}{g}$, $\frac{f'}{g'}$ are defined on (a, ∞). If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}\,.$$

□

Problem 2. (4pts) Find the limit $\lim_{x\to\infty} x\Big[\big(1+\frac{1}{x}\big)^x - e\Big]$ using L'Hôspital's rule.

Solution. Let $f(x) = e^{x\ln(1+\frac{1}{x})} - e$ and $g(x) = \frac{1}{x}$. Then $x\Big[\big(1+\frac{1}{x}\big)^x - e\Big] = \frac{f(x)}{g(x)}$, and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$. Note that

$$f'(x) = e^{x\ln(1+\frac{1}{x})}\Big[\ln\big(1+\frac{1}{x}\big) + x\cdot\frac{(-1)/x^2}{1+1/x}\Big] = \big[f(x)+e\big]\Big[\ln\big(1+\frac{1}{x}\big) - \frac{1}{1+x}\Big] \quad \text{and} \quad g'(x) = -\frac{1}{x^2}\,;$$

thus with $h(x) = \frac{1}{1+x} - \ln(1+\frac{1}{x})$ and $k(x) = \frac{1}{x^2}$, we have $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} k(x) = 0$ and

$$\frac{f'(x)}{g'(x)} = \big[f(x)+e\big]\frac{h(x)}{k(x)}\,.$$

Since $h'(x) = -\frac{1}{(1+x)^2} - \frac{(-1)/x^2}{1+1/x} = \frac{1}{x(x+1)^2}$ and $k'(x) = -2x^{-3}$, we have

$$\lim_{x\to\infty} \frac{h'(x)}{k'(x)} = -\lim_{x\to\infty} \frac{x^3}{2x(x+1)^2} = -\frac{1}{2}\,;$$

thus L'Hôspital's rule implies that $\lim_{x\to\infty} \frac{h'(x)}{k'(x)} = \lim_{x\to\infty} \frac{h(x)}{k(x)} = -\frac{1}{2}$ which, together with the fact that $\lim_{x\to\infty} f(x) = 0$, further implies that $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = -\frac{e}{2}$. Applying L'Hôspital's rule again, we conclude that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = -\frac{e}{2}$. □

Problem 3. (4pts) Find the indefinite integral $\int \frac{dx}{(4+x^2)^2}$.

Solution. Let $x = 2\tan u$. Then $dx = 2\sec^2 u\,du$; thus

$$\begin{aligned}
\int \frac{dx}{(4+x^2)^2} &= \int \frac{2\sec^2 u}{(4+4\tan^2 u)^2}\,du = \frac{1}{8}\int \frac{\sec^2 u}{\sec^4 u}\,du = \frac{1}{8}\int \cos^2 u\,du \\
&= \frac{1}{8}\int \frac{1+\cos 2u}{2}\,du = \frac{1}{16}\big(u + \frac{\sin 2u}{2}\big) + C = \frac{1}{16}\big(u + \sin u\cos u\big) + C \\
&= \frac{1}{16}\arctan\frac{x}{2} + \frac{1}{16}\cdot\frac{x}{2}\cdot\frac{1}{1+x^2/4} + C = \frac{1}{16}\arctan\frac{x}{2} + \frac{x}{8(4+x^2)} + C\,.
\end{aligned}$$

□